Time-domain matched interface and boundary (MIB) modeling of Debye dispersive media with curved interfaces

Duc Duy Nguyen, Shan Zhao

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, USA
Beijing Computational Science Research Center, Beijing 100084, PR China

Article history:
Received 11 October 2013
Received in revised form 14 July 2014
Accepted 19 August 2014
Available online 28 August 2014

Keywords:
Finite-difference time-domain (FDTD) Maxwell's equations Debye dispersive medium Transverse magnetic (TM) modes High order interface treatments Matched interface and boundary (MIB)

1. Introduction

Dispersive medium, in which the permittivity is a function of frequency [1–3], is often encountered in nature, such as in biological tissues, rocks, soils, ice, snow, and plasma. Due to such an omnipresent nature, the study of dispersive materials is indispensable to a wide range of electromagnetic applications. For instance, the ground penetrating radar (GPR) is a widely used nondestructive testing tool for subsurface sensing and detection of buried objects [4]. At the operating frequency range of the GPR, soil materials are known to exhibit strong frequency dispersion. Recently, microwave imaging has emerged as a promising alternative to the X-ray mammography for the early detection of breast cancer [5]. Similarly, the breast tissues are quite dispersive over the frequency band of interest in the microwave breast imaging. In these electromagnetic applications, to validate the experimental designs, the modeling of material dispersions is important for studying electromagnetic wave scattering and propagation in dispersive media. Moreover, in both the GPR and microwave breast imaging, there is usually a significant contrast in the dielectric properties between the target, such as mines or malignant tumors, and its surrounding. In practice, there is a pressing need to develop robust numerical tools to simulate how an electromagnetic wave interacts with a dispersive interface [6,7], which is defined to be the material boundary between a dispersive medium and a nondispersive medium or between two dispersive media.

* Corresponding author at: Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, USA. Tel.: +1 205 3485155; fax: +1 205 3487067.
E-mail address: szhao@ua.edu (S. Zhao).

http://dx.doi.org/10.1016/j.jcp.2014.08.038
0021-9991/© 2014 Elsevier Inc. All rights reserved.
In the computational electromagnetics (CEM), the dielectric interfaces are usually accommodated through defining body-fitted grids. Using unstructured grids or adaptive grids, the dispersive finite element time-domain methods (FETD) \([8–11]\) and dispersive discontinuous Galerkin time-domain (DGTD) methods \([12–14]\) are some of the most flexible methods for handling geometrically complex problems in the numerical solution of Maxwell’s equations over dispersive media. For dispersive finite-difference time-domain (FDTD) methods \([15–18]\) based on Cartesian grids or Yee meshes, great care has to be taken numerically to deal with the so-called staircasing approximation to arbitrarily curved interfaces. The accuracy reduction of the staircasing approximation can be partially recovered by using smoothing techniques \([19–21]\), in which the permittivities in the vicinity of an interface are averaged to form a smooth effective permittivity.

It is well known in the CEM literature that across dielectric interfaces, field components are nonsmooth or even discontinuous \([22,23]\). Thus, to achieve a higher order of accuracy, special interface treatments in which the interface conditions are properly imposed in the discretization, are indispensable. For nondispersive interfaces, many such high order interface treatments have been developed in the literature \([22–24]\). However, the generalization of such rigorous interface treatments to dispersive interfaces is extremely challenging, because a new difficulty is encountered in a dispersive interface problem. In particular, the wave solution will lose its regularity in a time variant manner across a dispersive interface \([6,7]\). In other words, the jump conditions at a dispersive interface are time dependent, whereas their counterparts for a nondispersive interface are time independent. The rigorous interface treatment for imposing time variant jump conditions is less well developed in the literature.

Recently, we have developed novel dispersive FDTD algorithms based on the matched interface and boundary (MIB) method \([23,24]\) for solving one-dimensional (1D) transverse electromagnetic systems \([6]\) and two-dimensional (2D) transverse magnetic (TM) systems \([7]\). An auxiliary differential equation (ADE) approach is employed in these dispersive FDTD methods \([6,7]\) to model the constitutive conditions. Moreover, the unsteady jump conditions are also reformulated by using the ADE approach into a form that can be rigorously incorporated in the FDTD discretization. High order convergences are numerically confirmed in solving 1D and 2D Maxwell’s equations. However, the previous dispersive MIB approaches \([6,7]\) can only handle straight interfaces and continuous solutions.

The generalization of the dispersive MIB methods to accommodate arbitrarily curved interfaces and discontinuous wave solutions is of practical importance, and is the main objective of this paper. In particular, we will develop a new MIB time-domain (MIBTD) method for solving 2D TM Maxwell’s equations with irregularly shaped dispersive interfaces based on a staggered Cartesian grid. Based on the ADE approach, the Debye dispersion model is coupled with the TM equations to derive interface auxiliary differential equations (IADEs) for describing the regularity changes in electromagnetic fields at dispersive interfaces. The enforcement of the resulting time dependent jump conditions fits the finite difference weights to the dispersive interfaces, in a sense to be made precise in Section 2.5, so that the staircasing error is eliminated. A uniformly second order of accuracy can be achieved in solving various dispersive interfaces. This paper is organized as follows. A set of Maxwell–Debye governing equations and the MIB algorithm are introduced in Section 2. The proposed method is illustrated through applications to several dispersive interfaces with different shapes in Section 3. Finally, this paper ends with a conclusion.

### 2. Theory and algorithm

In this section, a brief introduction to Maxwell theory and Debye dispersive model is first given. The jump conditions across a dispersive interface will be first established in terms of a local Cartesian coordinate. The rotation and decomposition of jump conditions into individual Cartesian directions will then be conducted. A new Maxwell–Debye system will be proposed. For a comparison, the application of the proposed formulation to a simpler situation with a nondispersive interface will also be considered. Finally, the details of the matched interface and boundary (MIB) treatments for coupled and uncoupled jump conditions, and smooth and nonsmooth interfaces will be offered.

#### 2.1. Maxwell theory and Debye model

Consider a linear isotropic electromagnetic structure involving dispersive materials. Assume that both the structure and incident wave are invariant in the \(z\) direction, so that all \(z\) derivatives of electromagnetic fields can be removed in Maxwell’s equations. Then, the resulting two-dimensional (2D) equations can be decomposed into two independent sets, i.e., the transverse magnetic (TM) mode and the transverse electric (TE) mode. In this work, we will focus only on the TM system in some inhomogeneous media

\[
\frac{\partial D_z}{\partial t} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}, \quad \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \frac{\partial E_z}{\partial x}, \quad \frac{\partial H_x}{\partial t} = -\frac{1}{\mu} \frac{\partial E_z}{\partial y},
\]

where \(E_z\) is the electric field component, \(H_x\) and \(H_y\) are, respectively, the magnetic field \(x\) and \(y\) components, and \(D_z\) is the electric displacement component.

The TM mode \((1)\) needs to be closed through the material constitutive equations. Here the magnetic constitutive relation is described as follows

\[
[B_x, B_y]^T = \mu [H_x, H_y]^T,
\]
where $B_x$ and $B_y$ are the magnetic flux density $x$ and $y$ components respectively, and $\mu$ is the magnetic permeability. In a nondispersive medium, the electric constitutive relation is given in the time domain,

$$D_z = \varepsilon E_z,$$

(3)

where $\varepsilon$ is the electric permittivity. However in a dispersive medium, the electric constitutive relation is prescribed in the frequency domain,

$$\tilde{D}_z = \tilde{\varepsilon}(\omega)\tilde{E}_z,$$

(4)

where the time-harmonic components are obtained via the Fourier transforms of the time-varying components. We will focus ourselves on the single-order Debye dispersion model in this work

$$\tilde{\varepsilon}(\omega) = \varepsilon_0 \left[ \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + i\omega\gamma} \right],$$

(5)

where $\varepsilon_0$, $\varepsilon_s$, and $\varepsilon_\infty$ are, respectively, the permittivities of free space, at static frequency, and at high frequency limit. Here $i$ is the imaginary unit, $\omega$ is the angular frequency, and $\gamma$ is the relaxation time constant.

The auxiliary differential equation (ADE) approach [15] can be used to represent the constitutive equations (4)-5 in the time domain

$$\gamma \frac{\partial D_z}{\partial t} + D_z = \varepsilon_0\varepsilon_\infty \frac{\partial E_z}{\partial t} + \varepsilon_0\varepsilon_s E_z,$$

(6)

so that a closed Maxwell system can be formed. There exist other closed Maxwell formulations for dispersive media based on the displacement or polarization current [10]. These formulations are essentially equivalent to the present ADE one via the introduction of an electric polarization field.

The complexity of a dispersive problem over a nondispersive one can be appreciated through governing equations. The nondispersive TM system consists of Eqs. (1) and (3), with parameters $\varepsilon$ and $\mu$ being possibly discontinuous at material interfaces. The electric displacement component $D_z$ is usually further eliminated to form a three-components system [23]. For the present Debye dispersive media, Eqs. (1) and (6) form a 2D closed TM system for four components $D_z$, $E_z$, $H_x$, and $H_y$. Across a general dispersive interface, four parameters, i.e., $\varepsilon_s$, $\varepsilon_\infty$, $\gamma$ and $\mu$, could be discontinuous. Moreover, since Eq. (6) involves time $t$, the resulting jump conditions at dispersive interfaces are also time dependent.

2.2. Interface jump conditions in a local coordinate

Consider a 2D domain of a rectangle shape $\Omega = [a, b] \times [c, d]$. An interface $\Gamma$ separates the domain $\Omega$ into two parts $\Omega^+$ and $\Omega^-$, see Fig. 1. In this paper, we will assume the interface $\Gamma$ being arbitrary curved and $C^0$ continuous. For two media on both hand sides of $\Gamma$, if one of them is a dispersive medium, we will refer to $\Gamma$ as a dispersive interface. For simplicity, we will assume the outside medium $\Omega^-$ being the air or vacuum in the present study. We note that the proposed formulation and algorithm can be applied to a more general dispersive interface with two dispersive materials.

Define the jump of a function $u(x, y)$ at $\Gamma$ to be $[u] = u^+ - u^-$. The superscript, $-$ or $+$, denotes the limiting value of function from one side or from the other side of the interface. For the present Debye-air dispersive interface, we assume the material equation (5) in the vacuum $\Omega^-$ as well with $\varepsilon_\infty^+ = \varepsilon_\infty^- = 1$. Then, the static frequency $\varepsilon_s$ and the high frequency limit $\varepsilon_\infty$ in (6) are both discontinuous across the interface $\Gamma$. Here, $\gamma^-$ is a free parameter so that we can assume $\gamma = \gamma^- = \gamma^+$ being a constant throughout $\Omega$. Moreover, the magnetic permeability $\mu$ could be discontinuous across interface $\Gamma$. In the positive side, i.e., free space, $\mu^+$ is assigned to be either 1 or $\mu_0$, depending on what type of Maxwell formalism is considered.

Across the $\Gamma$, field solutions in both media are related analytically via the jump conditions

$$\hat{n} \times (\hat{E}^+ - \hat{E}^-) = 0, \quad \hat{n} \cdot (\hat{D}^+ - \hat{D}^-) = 0, \quad \hat{n} \times (\hat{H}^+ - \hat{H}^-) = 0, \quad \hat{n} \cdot (\mu^+\hat{H}^+ - \mu^-\hat{H}^-) = 0,$$

(7)

where $\hat{n}$ is the unit vector normal to the interface, pointing from $\Omega^-$ into $\Omega^+$. We will consider jump conditions (7) in terms of a local Cartesian coordinate $(\hat{n}, \hat{r}, \hat{z})$, see Fig. 2. On such a local grid system, Eq. (7) gives rise to the following zeroth order jump conditions.
[\mathbf{E}_z] = 0, \quad [\mathbf{E}_\tau] = 0, \quad [\mathbf{D}_n] = 0, \quad [\mathbf{H}_z] = 0, \quad [\mathbf{H}_\tau] = 0, \quad [\mu\mathbf{H}_n] = 0. \quad (8)

First order jump conditions can be further derived based on the zeroth order conditions (8) and Maxwell’s equations [23,24].

For the \( E_z \) component, the first order conditions in terms of the local coordinate are quite simple

\[
\left[ \frac{1}{\mu} \frac{\partial E_z}{\partial n} \right] = 0, \quad \left[ \frac{\partial E_z}{\partial \tau} \right] = 0. \quad (9)
\]

For the magnetic components, three first order jump conditions can be similarly derived as in the nondispersive case [24]

\[
\left[ \frac{\partial H_\tau}{\partial \tau} \right] = 0, \quad \left[ \frac{\partial (\mu H_n)}{\partial n} \right] = 0, \quad \left[ \frac{\partial (\mu H_\tau)}{\partial \tau} \right] + \left[ \frac{\partial (\mu H_n)}{\partial n} \right] = 0. \quad (10)
\]

One more first order jump condition is required, which, however, is not readily available. We overcome this difficulty by using some interface auxiliary differential equations (IADEs) [6,7]. In particular, we assume \( [\mu D_\tau] = \psi(t,x,y) \) for some unknown function \( \psi(t,x,y) \). We multiply the first equation of (1) by \( \mu \) and take jump operations to it to attain the last jump condition

\[
\frac{\partial \psi(t,y)}{\partial t} = \left[ \frac{\partial (\mu D_z)}{\partial t} \right] = \left[ \frac{\partial (\mu H_x)}{\partial x} \right] - \left[ \frac{\partial (\mu H_y)}{\partial y} \right]. \quad (11)
\]

We note that the nonhomogeneous term in (11) is time dependent and unknown.

2.3. Interface jump conditions in the Cartesian coordinate

A key idea in the matched interface and boundary (MIB) interface treatment [23,24] is to decompose 2D jump conditions into one-dimensional (1D) ones along each Cartesian direction. We first consider the zeroth order jump conditions for Cartesian field components. For \( E_z \), we simply have \( [E_z] = 0 \) or \( E_z^+ - E_z^- = 0 \). In order to derive the jump conditions for \( H_x \) and \( H_y \), we use the following coordinate transformation

\[
H_n = \cos \theta H_x + \sin \theta H_y, \quad H_\tau = -\sin \theta H_x + \cos \theta H_y. \quad (12)
\]

where \( 0 \leq \theta < 2\pi \) is the angle between positive \( x \)-direction and the normal vector \( \mathbf{n} \), see Fig. 2. Such a transformation translates the zeroth order jump conditions \( [\mu H_n] = 0 \) and \( [H_\tau] = 0 \) into

\[
0 = [\mu H_n] = \cos \theta \mu^+ H_x^+ + \sin \theta \mu^+ H_y^+ - \cos \theta \mu^- H_x^- - \sin \theta \mu^- H_y^-, \quad (13)
\]

\[
0 = [H_\tau] = -\sin \theta H_x^+ + \cos \theta H_y^+ + \sin \theta H_x^- - \cos \theta H_y^- \quad (14).
\]

Eqs. (13) and (14) can be further simplified by eliminating negative components. We drop out the values from negative side since the incident wave propagates from the positive side. One will obtain, therefore, a new zeroth jump condition for \( H \) field by canceling \( H_x^- \) out but keeping \( H_y^- \) in (13) and (14). Another new zeroth jump condition will be obtained by eliminating \( H_y^- \) but keeping \( H_x^- \) in (13) and (14). These procedures give us the following jump conditions

\[
C_x^+ H_x^+ + C_y^+ H_y^+ - C_x^- H_x^- = 0, \quad (15)
\]

\[
D_x^+ H_x^+ + D_y^+ H_y^+ - D_x^- H_x^- = 0, \quad (16)
\]

where

\[
C_x^+ = (\mu^+ - \mu^-) \cos \theta \sin \theta, \quad C_y^+ = \sin^2 \theta \mu^+ + \cos^2 \theta \mu^- , \quad C_x^- = \mu^-, \quad C_y^- = \mu^-.
\]

\[
D_x^+ = \sin^2 \theta \mu^- + \cos^2 \theta \mu^+, \quad D_y^+ = (\mu^+ - \mu^-) \sin \theta \cos \theta \quad \text{and} \quad D_x^- = \mu^-.
\]
We then consider the first order jump conditions for Cartesian field components. To this end, we use the following derivative operators with respect to the local coordinate
\[
\frac{\partial}{\partial n} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \tau} = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}.
\] (17)

For the \(E_z\) component, we have
\[
0 = \left[ \frac{1}{\mu} \frac{\partial E_z}{\partial n} \right] = \frac{1}{\mu^+} \left( \cos \theta \left( \frac{\partial E_z}{\partial x} \right)^+ + \sin \theta \left( \frac{\partial E_z}{\partial y} \right)^+ \right) - \frac{1}{\mu^-} \left( \cos \theta \left( \frac{\partial E_z}{\partial x} \right)^- + \sin \theta \left( \frac{\partial E_z}{\partial y} \right)^- \right),
\] (18)
\[
0 = \left[ \frac{\partial E_z}{\partial \tau} \right] = -\sin \theta \left( \frac{\partial E_z}{\partial x} \right)^+ + \cos \theta \left( \frac{\partial E_z}{\partial y} \right)^+ - \left( -\sin \theta \left( \frac{\partial E_z}{\partial x} \right)^- + \cos \theta \left( \frac{\partial E_z}{\partial y} \right)^- \right). \tag{19}
\]

These first order jump conditions are similarly to the ones of 2D elliptic problem in [31]. Thus, through some simple linear combinations [31], we have the following three conditions for \(E_z\), including the aforementioned zeroth order one,
\[
E_z^+ - E_z^- = 0,
\]
\[
A_x^+ \left( \frac{\partial E_z}{\partial x} \right)^+ + A_y^+ \left( \frac{\partial E_z}{\partial y} \right)^+ - A_x^- \left( \frac{\partial E_z}{\partial x} \right)^- = 0,
\]
\[
B_x^+ \left( \frac{\partial E_z}{\partial x} \right)^+ + B_y^+ \left( \frac{\partial E_z}{\partial y} \right)^+ - B_y^- \left( \frac{\partial E_z}{\partial y} \right)^- = 0.
\] (20)

where \(A_x^+ = \frac{1}{\mu^+} \cos \theta + \frac{1}{\mu^-} \tan \theta \sin \theta, \quad A_y^+ = \frac{1}{\mu^+} - \frac{1}{\mu^-} \sin \theta, \quad A_x^- = \frac{1}{\mu^-} (\cos \theta + \tan \theta \sin \theta),\)
\[
B_x^+ = \frac{1}{\mu^+} \cos \theta, \quad B_y^+ = \frac{1}{\mu^-} \cos \theta \cot \theta + \frac{1}{\mu^+} \sin \theta, \quad B_y^- = \frac{1}{\mu^-} (\cos \theta \cot \theta + \sin \theta).
\]
These jump conditions are the same for both nondispersive and dispersive interfaces. When \(\mu\) is a constant, they can be simply handled by the previous interface schemes [7,24]. In a more complicated case, i.e., \(\mu\) is a piecewise constant, the classical MIB scheme [31] can successfully deal with them.

To attain the first order jump conditions for \(H_x\) and \(H_y\), we again employ the coordinate transformations defined in (12) and (17). For three conditions given in (10) and together with (11) based on the IADE, we have
\[
0 = \left[ \frac{\partial (\mu H_n)}{\partial n} \right] + \left[ \frac{\partial (\mu H_{\tau})}{\partial \tau} \right] = \left[ \frac{\partial (\mu H_x)}{\partial x} \right] + \left[ \frac{\partial (\mu H_y)}{\partial y} \right],
\] (21)
\[
0 = \left[ \frac{\partial (\mu H_{\tau})}{\partial \tau} \right] = \sin^2 \theta \left[ \frac{\partial (\mu H_x)}{\partial x} \right] - \cos \theta \sin \theta \left[ \frac{\partial (\mu H_x)}{\partial y} \right] - \sin \theta \cos \theta \left[ \frac{\partial (\mu H_y)}{\partial x} \right] + \cos^2 \theta \left[ \frac{\partial (\mu H_y)}{\partial y} \right]. \tag{22}
\]
\[
0 = \left[ \frac{\partial (\mu H_n)}{\partial \tau} \right] = -\sin \theta \cos \theta \left[ \frac{\partial (\mu H_x)}{\partial x} \right] + \cos^2 \theta \left[ \frac{\partial (\mu H_x)}{\partial y} \right] - \sin^2 \theta \left[ \frac{\partial (\mu H_y)}{\partial x} \right] + \cos \theta \sin \theta \left[ \frac{\partial (\mu H_y)}{\partial y} \right]. \tag{23}
\]

One can derive the first order jump conditions for \(H_x\) and \(H_y\) by solving the four equations (11), (21), (22) and (23)
\[
\frac{\partial (\mu H_y)}{\partial x} = \cos^2 \theta \frac{\partial \psi}{\partial t} - \sin(2\theta) \left[ \frac{\partial (\mu H_x)}{\partial y} \right], \tag{24}
\]
\[
\frac{\partial (\mu H_x)}{\partial x} = -\frac{\sin(2\theta)}{2} \frac{\partial \psi}{\partial t} - \cos(2\theta) \left[ \frac{\partial (\mu H_x)}{\partial \tau} \right], \tag{25}
\]
\[
\frac{\partial (\mu H_y)}{\partial y} = \frac{\sin(2\theta)}{2} \frac{\partial \psi}{\partial t} + \cos(2\theta) \left[ \frac{\partial (\mu H_x)}{\partial \tau} \right], \tag{26}
\]
\[
\frac{\partial (\mu H_x)}{\partial y} = -\sin^2 \theta \frac{\partial \psi}{\partial t} - \sin(2\theta) \left[ \frac{\partial (\mu H_x)}{\partial \tau} \right]. \tag{27}
\]

On the right sides of these conditions, while \(\psi(t, x, y)\) is time dependent and needs to be updated at each time step, the term \(\partial (\mu H_x)/\partial \tau\) can be further expanded by using the fact that \(\partial H_x/\partial \tau = 0\)
\[
\frac{\partial (\mu H_x)}{\partial \tau} = (\mu^+ - \mu^-) \left( \frac{\partial H_x}{\partial \tau} \right)^+ + \mu^- \left[ \frac{\partial H_x}{\partial \tau} \right]^-
= (\mu^+ - \mu^-) \left( \sin^2 \theta \left( \frac{\partial H_x}{\partial x} \right)^+ - \cos \theta \sin \theta \left( \frac{\partial H_x}{\partial y} \right)^+ - \sin \theta \cos \theta \left( \frac{\partial H_x}{\partial y} \right)^- + \cos^2 \theta \left( \frac{\partial H_x}{\partial y} \right)^+ \right). \tag{28}
\]
If $\mu$ is a piecewise constant across the interface, one can deduce another relation in $H$ field from the divergence free of magnetic flux density
\[ \frac{\partial H_x}{\partial x} = -\frac{\partial H_y}{\partial y} \tag{29} \]

Note that in the present implementation, Eq. (29) is only applied to the positive side. From (24)–(29), we attain the final first order jump conditions for $H_x$ and $H_y$

\[ C_{xy}^+ \frac{\partial H_x}{\partial y} + C_{yx}^+ \frac{\partial H_y}{\partial x} + C_{yy}^+ \frac{\partial H_y}{\partial y} + \partial \psi = C_{yx}^- \frac{\partial H_x}{\partial x}, \tag{30} \]

\[ D_{xx}^+ \frac{\partial H_x}{\partial x} + D_{xy}^+ \frac{\partial H_x}{\partial y} + D_{yx}^+ \frac{\partial H_y}{\partial x} + D_{yy}^+ \frac{\partial H_y}{\partial y} + \partial \psi = D_{xy}^- \frac{\partial H_x}{\partial y}, \tag{31} \]

\[ \hat{C}_{xy}^+ \frac{\partial H_x}{\partial y} + \hat{C}_{yx}^+ \frac{\partial H_y}{\partial x} + \hat{C}_{yy}^+ \frac{\partial H_y}{\partial y} + \hat{\partial \psi} = \hat{C}_{yx}^- \frac{\partial H_x}{\partial x}, \tag{32} \]

\[ \hat{D}_{xx}^+ \frac{\partial H_x}{\partial x} + \hat{D}_{xy}^+ \frac{\partial H_x}{\partial y} + \hat{D}_{yx}^+ \frac{\partial H_y}{\partial x} + \hat{D}_{yy}^+ \frac{\partial H_y}{\partial y} + \hat{\partial \psi} = \hat{D}_{xy}^- \frac{\partial H_x}{\partial y}, \tag{33} \]

where

\[ C_{xy}^+ = -\frac{\sin^2(2\theta)}{2}(\mu^+ - \mu^-), \quad C_{yx}^+ = \mu^+ - \frac{\sin^2(2\theta)}{2}(\mu^+ - \mu^-), \quad C_{yy}^+ = \mu^+, \]

\[ C_{xy}^- = \frac{\sin(4\theta)}{2}(\mu^+ - \mu^-), \quad C_{yx}^- = -\cos^2\theta, \quad C_{yy}^- = \mu^-, \]

\[ D_{xx}^+ = \mu^+ - \cos^2(2\theta)(\mu^+ - \mu^-), \quad D_{xy}^+ = -\frac{\sin(4\theta)}{4}(\mu^+ - \mu^-), \quad D_{yx}^+ = D_{xy}^-, \]

\[ D_{xy}^+ = \frac{\sin(2\theta)}{2}, \quad D_{yx}^- = \mu^-, \]

\[ \hat{C}_{xy}^+ = \frac{\sin(4\theta)}{4}(\mu^+ - \mu^-), \quad \hat{C}_{yx}^+ = \hat{\hat{C}}_{xy}^+, \quad \hat{C}_{yy}^+ = \mu^+ - \cos^2(2\theta)(\mu^+ - \mu^-), \]

\[ \hat{C}_{xy}^- = -\frac{\sin(2\theta)}{2}, \quad \hat{C}_{yx}^- = \mu^-, \]

\[ \hat{\partial \psi} = \frac{\sin(2\theta)}{2}(\mu^+ - \mu^-), \quad \hat{\partial \psi} = \frac{\sin(2\theta)}{2}(\mu^+ - \mu^-), \]

\[ \hat{D}_{xy}^+ = -(\mu^+ - \mu^-)\frac{\sin(4\theta)}{2}, \quad \hat{D}_{yx}^+ = \mu^+ - \frac{\sin^2(2\theta)}{2}(\mu^+ - \mu^-), \quad \hat{D}_{yx}^- = -\frac{\sin^2(2\theta)}{2}(\mu^+ - \mu^-), \]

\[ \hat{\partial \psi} = \sin^2\theta, \quad \hat{\partial \psi} = \sin^2\theta, \quad \text{and} \quad \hat{\partial \psi} = \mu^-. \]

Jump conditions (30) and (31) will be used for the MIB modeling of the magnetic fields along the $x$ direction. Jump conditions (32) and (33) will be applied for evaluating fictitious values of the magnetic fields in the $y$ direction.

On the other hand, when $\mu$ is a constant, i.e., $\mu^+ = \mu^-$, jump conditions for electric and magnetic fields can be greatly simplified. In particular, we have the following conditions for $E_z$

\[ [E_z] = 0, \quad \left[ \frac{\partial E_z}{\partial x} \right] = 0, \quad \left[ \frac{\partial E_z}{\partial y} \right] = 0. \tag{34} \]

Moreover, jump conditions for $H_x$ and $H_y$ are now decoupled

\[ [H_y] = \left[ \frac{\partial H_y}{\partial x} \right] = \frac{\partial \psi(t, x, y) \cos^2\theta}{\mu} \quad \text{(for $H_y$)}, \tag{35} \]

\[ [H_x] = \left[ \frac{\partial H_x}{\partial y} \right] = -\frac{\partial \psi(t, x, y) \sin^2\theta}{\mu} \quad \text{(for $H_x$)}. \tag{36} \]

We note that the jump conditions (35) and (36) are much simpler than their counterparts (30)–(33) in two senses. First, in (35) and (36), $x$ and $y$ directions are uncoupled, while in (30)–(33), $x$ and $y$ derivatives are coupled. Second, in (30)–(33), $H_x$ and $H_y$ are mixed in each condition, while conditions for $H_x$ and $H_y$ are unmixed in (35) and (36). Nevertheless, all these jump conditions are time dependent and share a common difficulty, i.e., there is an unknown nonhomogeneous term $\frac{\partial \psi}{\partial t}$, which has to be calculated at each time step. In order to deal with the time dependent, coupled and mixed jump conditions, new MIB methods have to be formulated.
2.4. Maxwell–Debye system and MIB time domain algorithm

In this subsection, a new Maxwell–Debye system will be constructed to facilitate the implementation of the proposed jump conditions. For the purpose of calculating \( \psi(t, x, y) \), we first multiply (6) with \( \mu \) and then conduct jump operations for it. This gives rise to the following IADE

\[
\gamma \frac{\partial \psi(t, x, y)}{\partial t} + \psi(t, x, y) = \epsilon_0 \gamma [\mu \epsilon_{\infty} \dot{E}_z] + \epsilon_0 [\mu \varepsilon_{z} E_z],
\]

where \( \dot{E}_z = \frac{\partial E_z}{\partial t} \). Since both \( E_z \) and \( \dot{E}_z \) are continuous across \( \Gamma \), the jump operations on the right hand side of (37) can be rewritten as

\[
g(t, x, y) := \epsilon_0 \gamma [\mu \epsilon_{\infty} \dot{E}_z] + \epsilon_0 [\mu \varepsilon_{z} E_z] = \epsilon_0 \gamma (\mu^+ \epsilon_{\infty}^+ - \mu^- \epsilon_{\infty}^-) \dot{E}_z + \epsilon_0 (\mu^+ \varepsilon_z^+ - \mu^- \varepsilon_z^-) E_z.
\]

Note that \( g(t, x, y) \) in (38) can be evaluated from the negative side in \( \Omega^- \) as well, i.e., based on \( \dot{E}_z^- \) and \( E_z^- \). The positive side is used in the present study, because the incident wave propagates from the positive domain \( \Omega^+ \). With \( g(t, x, y) \), (37) reduces to

\[
\gamma \frac{\partial \psi(t, x, y)}{\partial t} + \psi(t, x, y) = g(t, x, y)
\]

which can be regarded as an ordinary differential equation with respect to time \( t \), for given \( x \) and \( y \) grid locations.

In order to calculate the \( g(t, x, y) \) term in the IADE (39), we propose to utilize \( \dot{E}_z \) instead of \( D_z \) to formulate a modified Maxwell system for TM modes

\[
\frac{\partial E_z}{\partial t} = \dot{E}_z, \quad \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \frac{\partial E_z}{\partial x}, \quad \frac{\partial H_x}{\partial t} = -\frac{1}{\mu} \frac{\partial E_z}{\partial y},
\]

\[
\epsilon_0 \epsilon_{\infty} \gamma \frac{\partial \dot{E}_z}{\partial t} = -\epsilon_0 \varepsilon_z \dot{E}_z + \gamma \left( \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial E_z}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} \frac{\partial E_z}{\partial y} \right) \right) + \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right).
\]

(40)

It is noted in (40), we need to carry out the MIB interface treatment for \( E_z \) along both \( x \) and \( y \) directions. For two magnetic components, only one derivative is involved for each component, i.e., \( \frac{\partial H_y}{\partial x} \) and \( \frac{\partial H_x}{\partial y} \). Thus, if \( \mu \) is a constant, we will enforce \( x \) direction jump condition (35) and \( y \) direction jump condition (36), respectively, for \( H_y \) and \( H_x \). When \( \mu \) is a piecewise constant, a group of jump conditions including Eqs. (15), (16), (30) and (31) will be applied for \( H_y \). And we impose a group of jump conditions including Eqs. (15), (16), (32) and (33) for \( H_x \).

In our computations, a uniform staggered grid system shown in Fig. 3 is employed. As an extension of the standard Yee cell, the field components \( E_z \) and \( \dot{E}_z \) will be computed on the same collocation node, while \( H_x \) and \( H_y \) will be calculated on staggered positions. Without the loss of generality, we assume that the interface \( \Gamma \) will not pass any grid point in the present work. Away from the interface \( \Gamma \), the standard central finite differences are employed to discretize spatial derivatives in (40), while the finite difference weights in the vicinity of the interface shall be modified based on the MIB scheme, whose details will be offered in Section 2.5. The classical fourth order Runge–Kutta method [25] is employed to integrate both Maxwell–Debye system (40) and the IADE (39) with a fixed time increment \( \Delta t \), whereas other standard explicit or implicit time stepping methods may also be used for the time integration.

The proposed MIB time-domain (MIBTD) algorithm consists of three major stages in each fractional step of the Runge–Kutta time stepping. For simplicity, we will illustrate such a general procedure based on the explicit Euler scheme over the time interval \([t_k, t_{k+1}]\). This is because each fractional step of the four stage Runge–Kutta time integration can be regarded as an Euler integration [25]. In the following, we only consider the case with a constant magnetic permeability \( \mu \) in the present algorithm. The case with a piecewise constant \( \mu \) can be similarly treated. First, the jump conditions (34) will be imposed along both \( x \) and \( y \) directions for \( E_z \). For \( H_y \), the jump conditions (35) will be enforced along \( x \) direction based on known values of \( \frac{\partial \psi}{\partial x}(t_k, x_0, y_0) \) on some interface point \((x_0, y_0)\). Similarly, the MIB
Given $E_z, \dot{E}_z, H_z, H_y, g$, and $\frac{\partial \psi}{\partial t}$ values at $t_k$; Enforce (34) to determine fictitious values of $E_z$ at $t_k$; Enforce (35) to determine fictitious values of $H_y$ at $t_k$; Enforce (36) to determine fictitious values of $H_x$ at $t_k$.

Integrate (40) by the Euler scheme to attain $E_z, \dot{E}_z, H_z, H_y$ values at $t_{k+1}$; Integrate (39) by the Euler scheme to attain $\psi$ values at $t_{k+1}$.

Calculate $g$ values at $t_{k+1}$ by using (38); based on $E_z$ and $\dot{E}_z$ values at $t_{k+1}$; Calculate $\frac{\partial \psi}{\partial t}$ values at $t_{k+1}$ by using (39), based on $g$ values at $t_{k+1}$.

**Fig. 4.** A flowchart of the MIB algorithm based on the explicit Euler scheme over the time interval $[t_k, t_{k+1}]$ for the case of constant magnetic permeability $\mu$.

interface treatment will be conducted for $H_x$ along $y$ direction based on $\frac{\partial \psi}{\partial t}(t_k, x_0, y_0)$ values. With these interface treatments, all spatial derivatives involved in (40) can then be approximated by finite differences with a second order of accuracy. Second, four equations of (40) are time integrated to generate function values at $t_{k+1}$. Similarly, based on known $g(t_k, x_0, y_0)$ values, one integrates the IADE (39) for $\psi(t_{k+1}, x_0, y_0)$. Third, to facilitate computations of next step, one needs to calculate $\frac{\partial \psi}{\partial t}(t_{k+1}, x_0, y_0)$. Since now $\psi(t_{k+1}, x_0, y_0)$ is known, by using (39), we actually need to find $g(t_{k+1}, x_0, y_0) = \epsilon_0 \gamma(\mu^+ \epsilon_0^+ - \mu^- \epsilon_0^-) \dot{E}_z^+ (t_{k+1}, x_0, y_0) + \epsilon_0 (\mu^+ \epsilon_1^+ - \mu^- \epsilon_1^-) E_x^+ (t_{k+1}, x_0, y_0)$. Since $(x_0, y_0)$ is off-grid, $E_x^+ (t_{k+1}, x_0, y_0)$ and $\dot{E}_z^+ (t_{k+1}, x_0, y_0)$ are approximated via one-sided extrapolations based on several nearby function values of $E_z(t_{k+1}, x_i, y_j)$ and $\dot{E}_z(t_{k+1}, x_i, y_j)$ in the positive domain $\Omega^+$. Here $(x_i, y_j)$ are some nearby grid nodes. A flowchart summarizing these three stages of the MIB algorithm is shown in Fig. 4.

To benchmark the proposed Maxwell–Debye formulation and the dispersive MIBTD algorithm, we will first apply the MIBTD approach to a nondispersive interface problem with analytical solutions. Here, $\Omega^+$ and $\Omega^-$ are two different dielectric materials and the electric permittivity $\epsilon$ is discontinuous across the interface $\Gamma$. The governing equations of TM modes (40) now degenerate to

$$
\frac{\partial E_z}{\partial t} = \dot{E}_z, \quad \frac{\partial H_y}{\partial t} = \frac{1}{\mu} \frac{\partial E_z}{\partial x}, \quad \frac{\partial H_x}{\partial t} = -\frac{1}{\mu} \frac{\partial E_z}{\partial y},
$$

$$
\frac{\partial \dot{E}_z}{\partial t} = \frac{1}{\epsilon} \left( \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial E_z}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} \frac{\partial E_z}{\partial y} \right) \right),
$$

(41)

where no spatial derivatives are involved for $H_x$ and $H_y$. Thus, one only needs to carry out MIB interface treatment for $E_z$ along both Cartesian directions based on the jump conditions (20) or (34), depending on the continuous condition of $\mu$. The classical fourth order Runge–Kutta method is also employed to update Maxwell’s equations (41). This nondispersive MIBTD algorithm preserves essential features of the dispersive MIBTD algorithm.

2.5. MIB treatment for unmixed jump conditions

We first consider the MIB treatments for the case with $\mu$ being a constant. In this case, the jump conditions for electric and magnetic fields are unmixed ones, i.e., only one field component is involved in jump condition (34), (35) or (36). Moreover, these conditions are all one-dimensional ones. For simplicity, we demonstrate only the MIB enforcement of (34) for $E_z$ here. The jump conditions (35) and (36) for $H_y$ and $H_x$ can be similarly treated. We classify a Lipschitz continuous interface as a smooth one or a sharp one. Different MIB treatments are required to handle those interfaces.

2.5.1. Smooth interface

Here, a smooth interface is referred to a $C^1$ continuous curve with no sharp edge. The regular MIB method [23,24] will be used in this situation. The standard second order central finite difference approximation will be carried out away from
the interface. The finite difference weights of nodes in the vicinity of the interface, i.e., the so-called irregular points, shall be modified in order to satisfy jump conditions. A universal rule here is that to approximate function or its derivatives on one side of interface, one never directly refers to function values from the other side. Instead, in the MIB scheme, fictitious values from the other side of the interface will be used.

For example, suppose that \((x_i, y_j)\) belongs to \(\Omega^+\) and \((x_{i+1}, y_j)\) belongs to \(\Omega^-\). See Fig. 5. Denote the function value and fictitious value of \(E_z\) at the node \((x_i, y_j)\) as \(E_{i,j}\) and \(f_{i,j}\), respectively. The second order finite difference approximation to the double \(x\) derivative term in (40) will be modified to be

\[
\frac{\partial^2 E_z}{\partial x^2} \bigg|_{x_i, y_j} \approx \frac{E_{i-1,j} - 2E_{i,j} + f_{i+1,j}}{\Delta x^2}, \quad \frac{\partial^2 E_z}{\partial x^2} \bigg|_{x_{i+1}, y_j} \approx f_{i,j} - 2E_{i+1,j} + E_{i+2,j}.
\]

In these discretizations, the central difference approximation at a given node involves two neighboring nodes. For a smooth interface with a small curvature, one can usually guarantee that one of the two neighboring nodes is from the same subdomain as the center node. As a result, the standard MIB scheme is always applicable in this situation.

We determine two fictitious values, \(f_i\) and \(f_{i+1}\), by discretizing two jump conditions (34) in the same manner of Eq. (42), i.e., never referring to function values across the interface

\[
w_{0,1}^+ E_{i-1,j} + w_{0,2}^+ E_{i,j} + w_{0,3}^+ f_{i+1,j} = w_{0,1}^- E_{i,j} + w_{0,2}^- E_{i+1,j} + w_{0,3}^- E_{i+2,j} + [E]_{i,j},
\]

\[
w_{1,1}^+ E_{i-1,j} + w_{1,2}^+ E_{i,j} + w_{1,3}^+ f_{i+1,j} = w_{1,1}^- E_{i,j} + w_{1,2}^- E_{i+1,j} + w_{1,3}^- E_{i+2,j} + \left[\frac{\partial E}{\partial x}\right]_{i,j},
\]

where \(w_{k,l}^-\) and \(w_{k,l}^+\) for \(k = 1, 2\) and \(l = 0, 1, 3\) are one-sided finite difference weights, respectively, for left and right subdomains. Here the subscript \(l\) represents the interpolation \((l = 0)\) and the first order derivative approximation \((l = 1)\), and \(k\) is for grid index. One-sided approximations involving 2 grid nodes in one side are used to ensure a sufficiently high accuracy. Here \([E]_{i,j}\) and \(\left[\frac{\partial E}{\partial x}\right]_{i,j}\) are denoted as the zeroth and first order jump of \(E_z\) at interface node \((x_0, y_j)\). Both jump values are vanishing in the present work. However, we choose to present the MIB scheme in such a general way so that it can be readily applied to more complicated occasion.

Algebraically, Eqs. (43) and (44) can be translated equivalently into the following matrix system form

\[
W \cdot F = C \cdot U,
\]

where

\[
W = \begin{bmatrix} -w_{0,1}^- & w_{0,3}^+ \\ -w_{1,1}^- & w_{1,3}^+ \end{bmatrix}, \quad F = \begin{bmatrix} f_{i,j} \\ f_{i+1,j} \end{bmatrix}, \quad C = \begin{bmatrix} -w_{0,1}^- & -w_{0,2}^- & w_{0,3}^- & 1 & 0 \\ -w_{1,1}^- & -w_{1,2}^- & w_{1,3}^- & 0 & 1 \end{bmatrix}, \quad \text{and}\]

\[
U = \begin{bmatrix} E_{i-1,j} & E_{i,j} & E_{i+1,j} & E_{i+2,j} & [E]_{i,j} \end{bmatrix}, \quad \left[\frac{\partial E}{\partial x}\right]_{i,j}^T.
\]

Then the fictitious values \(F = [f_{i,j}, f_{i+1,j}]^T\) can be determined as follows

\[
F = W^{-1} \cdot C \cdot U.
\]

Formulation in (47) means that \(f_{i,j}\) and \(f_{i+1,j}\) are actually linear combinations in terms of the corresponding values \(E_{i-1,j}, E_{i,j}, E_{i+1,j}, E_{i+2,j}, [E]_{i,j}\) and \([\partial E/\partial x]_{i,j}\). The solved coefficients \(W^{-1} \cdot C\) will be referred as representation coefficients. By substituting \(f_{i,j}\) and \(f_{i+1,j}\) into Eq. (42), a second order finite difference discretization of the \(x\) derivative is constructed. The MIB treatment along the \(y\) direction can be similarly conducted. Finally, we note that the MIB interface matching needs only be carried out once at the beginning. The representation coefficients can then be used in all subsequent time steps [23].

Fig. 5. Illustration of the MIB grid partition for a smooth interface. Filled circles: grid nodes; Open circles: fictitious nodes; Square: interface node.
2.5.2. Sharp interface

Due to the presence of a sharp corner, we may not be able to guarantee that at least one of the two neighboring nodes involved in the central difference approximation is located in the same sub-domain of the center node. See Fig. 6. The same situation may still hold, even if the mesh is refined in both the x and y directions. Note that such a difficulty cannot be handled by the original MIB method [23,24], because it violates the universal rule of this method that the function value and its derivatives should be approximated on one side of the interface. Fortunately, this obstacle can be resolved by the double layer MIB scheme developed in [26].

To illustrate the idea, we suppose that a grid node \((x_i, y_j)\) belongs to the sub-domain \(\Omega^+\) while grid nodes \((x_{i-1}, y_j)\) and \((x_{i+1}, y_j)\) belong to the sub-domain \(\Omega^-\). See Fig. 6. Consequently, both neighboring nodes of \((x_i, y_j)\) along x-direction belong to a different sub-domain. Therefore, the discretization of \(E_z\) at \((x_i, y_j)\) involves only one main grid node and two fictitious nodes, whereas the discretizations of this field at \((x_{i-1}, y_j)\) or \((x_{i+1}, y_j)\) are still the same as in (42)

$$\frac{\partial^2 E_z}{\partial x^2} \bigg|_{x_{i-1}, y_j} \approx \frac{E_{i-2,j} - 2E_{i-1,j} + \hat{f}_{i,j}}{\Delta x^2}, \quad \frac{\partial^2 E_z}{\partial x^2} \bigg|_{x_{i+1}, y_j} \approx \frac{E_{i+2,j} - 2E_{i+1,j} + \hat{f}_{i,j}}{\Delta x^2}.$$

\(48\)

Note that, \(\hat{f}_{i,j}\) and \(\hat{f}_{i,j}\) are two fictitious points located at the same grid node \((x_i, y_j)\) and they could take different values. Similarly to the original MIB method, fictitious values are determined by discretizing two jump conditions (34). In the traditional MIB, these jump conditions are imposed only at one interface location for solving two fictitious values. But the double layer MIB method [26] will enforce the jump conditions at two consecutive interface locations for the purpose of evaluating four fictitious values at the same time. Referring to Fig. 6, two jump conditions (34) are discretized at two adjacent interface positions \((x_0, y_j)\) and \((x_2, y_j)\)

\(49\)

$$\begin{align*}
\hat{w}_{0}^{+} E_{i-2,j} + \hat{w}_{0}^{-} E_{i-1,j} + \hat{w}_{0}^{-} f_{i,j} &= \hat{w}_{0}^{+} E_{i+1,j} + \hat{w}_{0}^{-} E_{i-1,j} + \hat{w}_{0}^{-} f_{i,j} - \bar{E}_{0,j-1}, \\
\hat{w}_{1}^{+} E_{i-2,j} + \hat{w}_{1}^{-} E_{i-1,j} + \hat{w}_{1}^{-} f_{i,j} &= \hat{w}_{1}^{+} E_{i+1,j} + \hat{w}_{1}^{-} E_{i-1,j} + \hat{w}_{1}^{-} f_{i,j} - \bar{E}_{0,j}. \\
\hat{w}_{1}^{+} E_{i-2,j} + \hat{w}_{1}^{-} E_{i-1,j} + \hat{w}_{1}^{+} f_{i,j} &= \hat{w}_{1}^{+} E_{i+1,j} + \hat{w}_{1}^{-} E_{i+1,j} + \hat{w}_{1}^{-} f_{i,j} + \bar{E}_{0,j}. \\
\hat{w}_{1}^{+} E_{i-2,j} + \hat{w}_{1}^{-} E_{i-1,j} + \hat{w}_{1}^{-} f_{i,j} &= \hat{w}_{1}^{+} E_{i+1,j} + \hat{w}_{1}^{-} E_{i+1,j} + \hat{w}_{1}^{-} f_{i,j} + \bar{E}_{0,j}. \\
\end{align*}$$

where the finite difference weights \(\hat{w}_{0}^{+}, \hat{w}_{0}^{-}, \hat{w}_{1}^{+}, \hat{w}_{1}^{-}\) for \(k = 1, 2\) and \(l = 0, 1\) are defined similarly as in the smooth interface case. Here \([\bar{E}_{0,j}], \frac{\partial E}{\partial x} l_{0,j}, \frac{\partial E}{\partial x} l_{0,j}, \frac{\partial E}{\partial x} l_{0,j} l_{0,j}\) are denoted as the zeroth and first order jumps of \(E_z\) at interface nodes \((x_0, y_j)\) and \((x_2, y_j)\). We again use the matrix language to translate Eqs. (49), (50), (51) and (52) into the following form

$$W \cdot F = C \cdot U,$$

\(53\)

where

\[ W = \begin{bmatrix} -w_{0,1} & \hat{w}_{0,1} & 0 & -w_{0,3} \\ -w_{1,1} & \hat{w}_{1,1} & 0 & -w_{1,3} \\ w_{0,1} & 0 & -\hat{w}_{0,1} & \hat{w}_{0,3} \\ w_{1,1} & 0 & -\hat{w}_{1,1} & \hat{w}_{1,3} \end{bmatrix}, \quad F = \begin{bmatrix} f_{i-1,j} \\ \hat{f}_{i,j} \\ \hat{f}_{i,j} \\ f_{i+1,j} \end{bmatrix}, \]

\[ C = \begin{bmatrix} \bar{E}_{0,j} \\ \frac{\partial E}{\partial x} l_{0,j} \\ \frac{\partial E}{\partial x} l_{0,j} \\ \frac{\partial E}{\partial x} l_{0,j} \end{bmatrix}. \]
Fig. 7. Illustration of the MIB grid partition for the case of a piecewise constant $\mu$. Filled circles: grid nodes for $H_x$; Fill triangles: nodes for $H_y$; Open circles: fictitious nodes for $H_y$; Open triangles: fictitious nodes for $H_x$; Open squares: auxiliary nodes for $H_x$ and $H_y$; Filled square: interface node.

\[
C = \begin{bmatrix}
-\tilde{w}_{0,1}^- & -\tilde{w}_{0,2}^- & w_{0,2}^+ & 0 & 0 & -1 & 0 & 0 & 0 \\
-\tilde{w}_{1,1}^- & -\tilde{w}_{1,2}^- & w_{1,2}^+ & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -\tilde{w}_{0,2}^+ & \hat{w}_{0,2}^- & \tilde{w}_{0,3}^- & 0 & 0 & 1 \\
0 & 0 & -\tilde{w}_{1,2}^+ & \hat{w}_{1,2}^- & \tilde{w}_{1,3}^- & 0 & 0 & 1
\end{bmatrix},
\]

and

\[
U = \left[E_{i-1,j}, E_{i-1,j}, E_{i,j}, E_{i+1,j}, E_{i+2,j}, [E]_{0,j}, \left[\frac{\partial E}{\partial x}\right]_{0,j}, [E]_{0,j}, \left[\frac{\partial E}{\partial x}\right]_{0,j}\right]^T.
\]

Then fictitious values can be easily found as follows

\[
F = W^{-1} \cdot C \cdot U.
\]

Eq. (55) implies that the fictitious values are the linear combinations of the function values and physical jumps. With these expansions of the fictitious values, one can discretize the second derivative of $E_2$ in (48) with a second order of accuracy. The MIB treatment for the $y$ grid line can be similarly carried out.

2.6. MIB treatment for mixed jump conditions

We next consider the case when $\mu$ is a piecewise constant. The jump conditions of electric and magnetic fields are now the mixed ones, so that the fictitious value determination of $H_x$ and $H_y$ has to conducted simultaneously. Thus, a new MIB method that differs from the one proposed in the previous subsection, has to be developed. In the present study, it is sufficient to illustrate this MIB method in the $x$ direction by discretizing Eqs. (15), (16), (30) and (31). The $y$ direction treatment can be similarly formed.

In the MIB grid partition shown in Fig. 7, $(x_0, y_0)$ is the location of the interface point. Suppose that both nodes $(x_{i-1/2}, y_j)$ and $(x_i, y_j)$ belong to $\Omega^-$ and node $(x_{i+1/2}, y_j)$ belongs to $\Omega^+$. By utilizing the same notations defined above, $H_y^{i+1/2,j}$ and $f_y^{i+1/2,j}$ are denoted as function value and fictitious value of $H_y$ at node $(x_{i+1/2}, y_j)$ respectively. Similar notations can be used for $H_x$ by changing subscript $y$ by $x$. The $x$ derivative of $H_y$ at irregular node $(x_i, y_j)$ in Fig. 7 should be modeled as follows

\[
\frac{\partial H_y}{\partial x} \bigg|_{(x_i, y_j)} \approx \frac{f_y^{i+1/2,j} - H_y^{i-1/2,j}}{\Delta x}.
\]

To seek four fictitious values $f_y^{i-1/2,j}$, $f_y^{i+1/2,j}$, $f_y^{i+1,j}$ and $f_y^{i+1,j}$ for the present case, we first discretize the $x$ derivatives in four jump conditions (15), (16), (30) and (31) at interface position $(x_0, y_0)$

\[
C_x^+ \left( \tilde{w}_{0,1}^+ f_y^{i,j} + \tilde{w}_{0,1}^+ H_y^{i+1,j} + \tilde{w}_{0,1}^+ H_y^{i+2,j} \right) + C_y^+ \left( \frac{w^+}{0,i-\frac{1}{2}} f_y^{i-\frac{1}{2},j} + \frac{w^+}{0,i+\frac{1}{2}} H_y^{i+,j} + \frac{w^+}{0,i+\frac{1}{2}} H_y^{i+,j} \right) \\
= C_y^- \left( \tilde{w}_{0,1}^- H_y^{i-\frac{1}{2},j} + \tilde{w}_{0,1}^- H_y^{i-\frac{1}{2},j} + \tilde{w}_{0,1}^- f_y^{i+\frac{1}{2},j} \right),
\]

\[
D_x^+ \left( \tilde{w}_{0,1}^+ f_y^{i,j} + \tilde{w}_{0,1}^+ H_y^{i+1,j} + \tilde{w}_{0,1}^+ H_y^{i+2,j} \right) + D_y^+ \left( \frac{w^+}{0,i-\frac{1}{2}} f_y^{i-\frac{1}{2},j} + \frac{w^+}{0,i+\frac{1}{2}} H_y^{i+,j} + \frac{w^+}{0,i+\frac{1}{2}} H_y^{i+,j} \right)
\]
Similarly, this can be evaluated at the interface node \((x_0, y_0)\) by using the following relation

\[
(H_{0,j}^y)^+ = -\frac{C^+}{C^+} (H_{0,j}^y)^- + \frac{C^-}{C^-} (H_{0,j}^y)^-
= -\frac{C^+}{C^+} (\tilde{w}_{0\rightarrow i+1,j}^x + \tilde{w}_{0\rightarrow i+1,j}^x) + \frac{C^-}{C^-} (\tilde{w}_{0\rightarrow i-1,j}^x + \tilde{w}_{0\rightarrow i+1,j}^x)
+ \frac{C^-}{C^-} (\tilde{w}_{0\rightarrow i+1,j}^x + \tilde{w}_{0\rightarrow i+1,j}^x),
\]

where the finite difference weights \(\tilde{w}_{i,k}^x\), \(\tilde{w}_{i,k}^y\), \(\tilde{w}_{i,k}^x\), and \(\tilde{w}_{i,k}^y\), are defined similarly as in the previous studies.

We then discretize the remaining \(y\) derivatives which are \((\partial H_y/\partial y)^+\) and \((\partial H_y/\partial y)^-\). We here only provide the detail of discretization of term \((\partial H_y/\partial x)^+\) since the \(y\) derivative of \(H_x\) can be similarly evaluated. Approximation of \((\partial H_y/\partial y)^+\) at the interface node \((x_0, y_0)\) requires three auxiliary points \((x_0, y_0)\), \((x_0, y_{j+1})\) and \((x_0, y_{j+2})\) on the positive side.

\[
\left. \left(\frac{\partial H_y}{\partial y}\right)^+ \right|_{x_0,y_0} \approx p_{1,j}^+ (H_{0,j}^y)^+ + p_{1,j}^+ H_{0,j+1}^y + p_{1,j+2}^+ H_{0,j+2}^y.
\]

Some further approximations have to be conducted since these auxiliary values are not determined on the grid nodes. Values \(H_{0,j+1}^y\) and \(H_{0,j+2}^y\) could be interpolated by using grid nodes \((x_{i-1/2}, y_{j+1})\), \((x_{i+1/2}, y_{j+1})\), \((x_{i+1/2}, y_{j+1})\) and \((x_{i-1/2}, y_{j+2})\) respectively. Refer to Fig. 7 for the locations of these grid nodes. Since \((H_{0,j}^y)^+\) is evaluated on the interface point \((x_0, y_0)\), its interpolation can be derived by using the following relation

\[
(H_{0,j}^y)^+ = -\frac{C^+}{C^+} (H_{0,j}^y)^- + \frac{C^-}{C^-} (H_{0,j}^y)^-
= -\frac{C^+}{C^+} (\tilde{w}_{0\rightarrow i+1,j}^x + \tilde{w}_{0\rightarrow i+1,j}^x) + \frac{C^-}{C^-} (\tilde{w}_{0\rightarrow i-1,j}^x + \tilde{w}_{0\rightarrow i+1,j}^x)
+ \frac{C^-}{C^-} (\tilde{w}_{0\rightarrow i-1,j}^x + \tilde{w}_{0\rightarrow i+1,j}^x).
\]

When all the \(x\) and \(y\) derivatives of the related fields are explicitly discretized, one can algebraically transform Eqs. (57)–(60) to the following form

\[
W \cdot F = C \cdot U,
\]

where

\[
W = [w_1, w_2, w_3, w_4]^\top \in \mathbb{R}^{4 \times 4}, \quad F = [f_x^{i+1/2,j}, f_y^{i+1/2,j}, f_x^{i+1/2,j}, f_y^{i+1/2,j}]^\top \in \mathbb{R}^{4 \times 1},
\]

\[
C = [c_1, c_2, c_3, c_4]^\top \in \mathbb{R}^{4 \times 12},
\]

and

\[
U = [H_x^{i+1/2,j}, H_y^{i+1/2,j}, H_x^{i+1/2,j}, H_y^{i+1/2,j}, H_x^{i+1/2,j}, H_y^{i+1/2,j}, H_x^{i+1/2,j}, H_y^{i+1/2,j}, H_x^{i+1/2,j}, H_y^{i+1/2,j}, H_x^{i+1/2,j}, H_y^{i+1/2,j}]^\top \in \mathbb{R}^{12 \times 1}.
\]

Note that vectors \(W\) and \(C\) can be properly determined through some algebraic steps. These calculations are tedious and will not be provided here to save space.

At this stage, the fictitious values \(F\) can be easily found by utilizing the following relation

\[
F = W^{-1} \cdot C \cdot U.
\]

Similarly to the above MIB treatments, the fictitious values \(F\) are the linear combination of function values and jump values in \(U\). Once again, the representation coefficients in \(W^{-1} \cdot C\) are only computed once at the beginning. They are reused for the MIB treatments in all subsequent time steps.
3. Numerical experiments

In this section, we validate the proposed matched interface and boundary time-domain (MIBTD) method by considering nondispersive and dispersive interface problems with constant curvature, nonconstant curvatures, and sharp corners. In all studies, a uniform mesh is employed for the \( E_z \) component, in which the numbers of grid points in both \( x \) and \( y \) directions are the same, i.e., \( N_x = N_y \). For simplicity, we denote \( N = N_x = N_y \). The staggered grids for \( H_x \) and \( H_y \) can then be correspondingly formed. By setting the initial time as \( t = 0 \), numerical time integration is carried out for \( N_t \) steps, until a stopping time \( T \). Here the time increment is given as \( \Delta t = T/N_t \).

3.1. Nondispersive cases: circular dielectric cylinder

We first consider a nondispersive interface problem with analytical solutions. This enables us to quantitatively analyze the performance of the proposed nondispersive MIBTD algorithm, which attains many features of the proposed dispersive MIBTD approach. We note that for more complicated dispersive interface problems, no closed form analytical solutions are available.

Following the literature work [27], the dimensionless form of the hybrid Maxwell’s system (41) is considered, i.e., \( \epsilon_0 = \mu_0 = 1 \) in the air. In a computational domain \( \Omega = [−1, 1]^2 \), a dielectric cylinder in the free space is studied with the center at \((0, 0)\) and the radius being \( r_0 \). The interface is defined as \( \Gamma : \sqrt{x^2 + y^2} = r_0 \). The subdomain \( \Omega^− \) is inside the cylinder with assigned relative dielectric parameters \( \mu_2, \epsilon_2 \), while the subdomain \( \Omega^+ \) consists of the region outside the cylinder, which is assumed to be vacuum with the relative dielectric parameters \( \mu_1, \epsilon_1 \). Here we take \( \mu_1 = \mu_2 = \epsilon_1 = 1 \). The transverse magnetic (TM) electromagnetic wave is scattered by a time-harmonic incident plane unit wavelength wave of the form [27]

\[
E^{\text{inc}}_x = e^{-i(k_1 x + \omega t)}, \quad H^{\text{inc}}_y = -e^{-i(k_1 x + \omega t)},
\]

where \( k_1 = \omega/\sqrt{\mu_1 \epsilon_1} \) is the propagation constant in the free-space medium. Then the exact solution for such incident waves is computed by using the summed-series technique [27,28] as follows

\[
E_x(x, y, t) = E_x(r, \theta, t) = e^{-i\omega t} \left\{ \begin{array}{cl}
\sum_{n=-\infty}^{\infty} A_n J_n(k_2 r) e^{i\omega \theta}, & r \leq r_0, \\
\sum_{n=-\infty}^{\infty} (i^n J_n(k_2 r) + B_n H^{(2)}_n(k_2 r)) e^{i\omega \theta}, & r > r_0,
\end{array} \right.
\]

where \((r, \theta)\) is the polar coordinate at \((x, y)\), \( k_2 \) is the propagation constant for the dielectric cylinder, \( J_n \) and \( H^{(2)}_n \) represent, respectively, the nth order Bessel function of the first kind and the Hankel function of the second kind, and the wave amplitude coefficients equal to

\[
A_n = i^{-n} \left( \frac{k_1}{\mu} \right) J'_n(k_1 r_0) H^{(2)}_n(k_1 r_0) - (k_1/\mu) H^{(2)}_n(k_1 r_0) J_n(k_1 r_0),
\]

\[
B_n = i^{−n} \left( \frac{k_1}{\mu} \right) J'_n(k_2 r_0) H^{(2)}_n(k_2 r_0) - (k_2/\mu) H^{(2)}_n(k_2 r_0) J_n(k_2 r_0).
\]

For the purpose of finding analytical forms for \( H_x \) and \( H_y \), we rewrite Eqs. (40) into the polar coordinate

\[
\frac{\partial H_x}{\partial t} = -\frac{1}{\mu \sigma} \frac{\partial E_x}{\partial \theta}, \quad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu \sigma} \frac{\partial E_x}{\partial r}.
\]

Then the radial component and angular component of the total magnetic field are given as

\[
H_r(r, \theta, t) = e^{-i\omega t} \left\{ \begin{array}{cl}
\sum_{n=-\infty}^{\infty} i^{−n} A_n J_n(k_2 r) e^{i\omega \theta}, & r \leq r_0, \\
\sum_{n=-\infty}^{\infty} i^n (i^n J_n(k_1 r) + B_n H^{(2)}_n(k_1 r)) e^{i\omega \theta}, & r > r_0,
\end{array} \right.
\]

\[
H_\theta(r, \theta, t) = e^{-i\omega t} \left\{ \begin{array}{cl}
\sum_{n=-\infty}^{\infty} i^{−n} A_n' J'_n(k_2 r) e^{i\omega \theta}, & r \leq r_0, \\
\sum_{n=-\infty}^{\infty} i^n (i^n J'_n(k_1 r) + B_n H^{(2)}_n'(k_1 r)) e^{i\omega \theta}, & r > r_0.
\end{array} \right.
\]

By using the coordinate transformation, exact solutions for \( H_x \) and \( H_y \) are determined as follows

\[
H_x = \cos \theta H_r - \sin \theta H_\theta, \quad H_y = \sin \theta H_r + \cos \theta H_\theta.
\]

In our computations, two nondispersive cases with different dielectric materials for the cylinder, i.e., \( \epsilon_2 = 2.25 \) and \( \epsilon_2 = 10 \), are studied. In both cases, we assign the radius of the cylinder to be \( r_0 = 0.6 \), the angular frequency to be \( \omega = 2\pi \), and the stopping time to be \( T = 1 \). We will set the mesh size \( N \) as an even integer so that a possible singularity of the Bessel functions at the origin \((0,0)\) can be avoided. The initial solution and boundary conditions are imposed according to
the exact solution. Besides the proposed non-dispersive MIBTD algorithm, a finite difference time-domain (FDTD) method is also considered, which solves the original TM equations (1) with the material equation (3) being substituted into. The same spatial and temporal discretizations as in the proposed MIBTD method, i.e., the central finite difference discretization on the staggered Yee grid and the fourth order Runge–Kutta time integration, are employed in this FDTD method. Numerically, the major difference between the present FDTD and MIBTD algorithms is that there is no matched interface and boundary (MIB) interface treatment being conducted in the FDTD method.

We first investigate the stability of the FDTD and MIBTD methods. For this purpose, we consider the case with \( \epsilon_2 = 2.25 \) and \( T = 1 \). Other cases with \( \epsilon_2 = 10 \) and/or different \( T \) values give essentially the same results. It is well known that the general Courant–Friedrichs–Lewy (CFL) stability condition for solving Maxwell’s equations can be given as [23]

\[
\Delta t \leq C = \frac{\Delta x}{v_{\text{max}} \sqrt{d}}
\]

where \( d = 2 \) is the dimension number, and the maximum velocity of the propagating wave is defined as \( v_{\text{max}} = \max\{\frac{1}{\sqrt{\mu_1}}, \frac{1}{\sqrt{\mu_2}}\} \). For \( \mu_1 = \mu_2 = \epsilon_1 = 1 \) and \( \epsilon_2 = 2.25 \) or 10, we have \( v_{\text{max}} = 1 \) for this dimensionless Maxwell system. The value of the CFL constant \( C \) is determined by both the spatial and temporal discretizations. We note that because the present FDTD approach is based on the fourth order Runge–Kutta method, instead of the Leapfrog scheme, the present CFL constant \( C \) differs from the one of the standard FDTD scheme by a factor of \( \sqrt{2} [23] \). In the present study, it is interesting to numerically detect this constant by studying a critical \( N_t \) value for a given \( N \) value. Here, the critical \( N_t \) value is defined in the sense that an even smaller \( N_t \) value will render the computation to be unstable. In particular, for a given \( N \) value, we examine all possible \( N_t \) values. For a particular \( N_t \) value, say \( N_t^* \), such that the final \( L_\infty \) error is less than a tolerance, e.g., 1, the computation is said to be stable. If the final error of \( N_t^* - 1 \) is greater than or equal to the tolerance, the empirical critical \( N_t \) value is taken as \( N_t^* \). By considering several \( N \) values, the critical \( N_t \) values and the corresponding CFL constants for both the FDTD and MIBTD methods are reported in Table 1. It can be seen that as \( \Delta x \to 0 \), the CFL number \( C \) of the present FDTD method converges to its analytical value \( C = \sqrt{2} [23] \). This validates the present procedure for the numerical stability analysis. For the proposed MIBTD algorithm, no analytical result is available, due to the sophisticated MIB interface treatments. Numerically, the CFL number is found to be a constant within the interval \( C \in [1.21, 1.26] \). In other words, the MIBTD method satisfies the same CFL condition as the FDTD method, with a slightly smaller CFL number.

We next examine the accuracy of the FDTD and MIBTD methods. To ensure the stability, we take the CFL constant to be \( C \approx 0.7 \) in both methods. The maximum errors in \( E_z \) of the FDTD and MIBTD methods are listed in Table 2 for both \( \epsilon_2 = 2.25 \) and \( \epsilon_2 = 10 \). It can be seen that the MIBTD scheme achieves a uniformly second order of accuracy in both cases, while the orders of the FDTD method are degraded. For a visual comparison, the \( L_\infty \) errors of two schemes are also depicted in Fig. 8. In all cases, the numerical errors based on different mesh size \( N \) are plotted as dashed lines. A linear least-squares fitting [24] is then conducted in the log–log scale. The fitted convergence lines are shown as solid lines in Fig. 8. Moreover, the fitted slope \( s \) essentially represents the overall numerical order of the scheme. These results clearly indicate that the MIBTD method achieves the second order convergence while the global accuracy of the traditional FDTD algorithms is reduced to first order because of the loss of regularities in wave solutions near the interface. The contour plots and slice plots of the MIBTD solutions are shown in Fig. 9 and Fig. 10, respectively, for \( \epsilon_2 = 2.25 \) and \( \epsilon_2 = 10 \). The slices plots obviously reveal that the \( E_z \) is \( C^1 \) continuous across the interface but \( H_y \) and \( H_z \) are only \( C^0 \). A comparison between Fig. 9 and Fig. 10 indicates that a highly oscillatory solution is involved in the case \( \epsilon = 10 \), especially inside the dielectric cylinder. Only high order methods, such as the proposed MIBTD algorithm, can capture such high frequency solutions accurately.

### Table 1
Critical CFL number \( C \) for the non-dispersive interface problem.

<table>
<thead>
<tr>
<th>( N )</th>
<th>FDTD</th>
<th>MIBTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Critical ( N_t )</td>
<td>( C )</td>
<td>Critical ( N_t )</td>
</tr>
<tr>
<td>100</td>
<td>49</td>
<td>1.4431</td>
</tr>
<tr>
<td>200</td>
<td>98</td>
<td>1.4431</td>
</tr>
<tr>
<td>400</td>
<td>198</td>
<td>1.4285</td>
</tr>
<tr>
<td>800</td>
<td>398</td>
<td>1.4213</td>
</tr>
</tbody>
</table>

### Table 2
Numerical convergence test of the non-dispersive interface problem in the \( L_\infty \) norm with \( \epsilon_2 = 2.25 \) and \( \epsilon_2 = 10 \).

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \epsilon_2 = 2.25 )</th>
<th>( \epsilon_2 = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDTD</td>
<td>MIBTD</td>
<td>FDTD</td>
</tr>
<tr>
<td>Error</td>
<td>Order</td>
<td>Error</td>
</tr>
<tr>
<td>320</td>
<td>5.70e−3</td>
<td>1.20e−3</td>
</tr>
<tr>
<td>400</td>
<td>4.50e−3</td>
<td>7.70e−4</td>
</tr>
<tr>
<td>640</td>
<td>2.30e−3</td>
<td>1.43</td>
</tr>
<tr>
<td>800</td>
<td>2.10e−3</td>
<td>0.41</td>
</tr>
</tbody>
</table>
Fig. 8. Numerical convergence tests for the nondispersive interface problem. (a) $\epsilon_2 = 2.25$; (b) $\epsilon_2 = 10$. In both charts, the numerical errors are plotted as dashed lines, while the least-squares fitted convergence lines are shown as solid lines.

Fig. 9. The MIBTD solution with $N = 320$ at $t = 1$ for the nondispersive cylinder with $\epsilon = 2.25$. (a) Contour plot of $E_z$. (b) $E_z(x, 0)$. (c) Contour plot of $H_y$. (d) $H_y(x, 0)$. (e) Contour plot of $H_x$. (f) $H_x(0, y)$. 
Fig. 10. The MIBTD solution with $N = 320$ at $t = 1$ for the nondispersive cylinder with $\epsilon = 10$. (a) Contour plot of $E_z$. (b) $E_z(x, 0)$. (c) Contour plot of $H_y$. (d) $H_y(x, 0)$. (e) Contour plot of $H_x$. (f) $H_x(0, y)$.

Table 3
CPU time in seconds of the FDTD method ($T_F$) and the MIBTD method ($T_M$) for the nondispersive interface problem.

<table>
<thead>
<tr>
<th>$N$</th>
<th>CPU time of FDTD ($T_F$)</th>
<th>CPU time of MIBTD ($T_M$)</th>
<th>$T_M/T_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1.47</td>
<td>2.22</td>
<td>1.51</td>
</tr>
<tr>
<td>160</td>
<td>5.75</td>
<td>8.45</td>
<td>1.47</td>
</tr>
<tr>
<td>200</td>
<td>9.33</td>
<td>14.77</td>
<td>1.58</td>
</tr>
<tr>
<td>320</td>
<td>33.33</td>
<td>57.41</td>
<td>1.72</td>
</tr>
<tr>
<td>400</td>
<td>59.48</td>
<td>109.31</td>
<td>1.84</td>
</tr>
<tr>
<td>640</td>
<td>226.61</td>
<td>414.69</td>
<td>1.83</td>
</tr>
<tr>
<td>800</td>
<td>424.48</td>
<td>756.77</td>
<td>1.78</td>
</tr>
</tbody>
</table>

In our last test of the nondispersive interface problem, we report the CPU time in seconds consumed by the FDTD and MIBTD computations in Table 3. Here, all computations are conducted on an Intel i5-2520M CPU core operating at 2.5 GHz and 6 GB of memory. Like in the accuracy test, we also take the CFL constant to be $C = 0.7$ in both approaches. It can be observed that the MIBTD method costs more CPU time than the FDTD method, due to the additional numerical treatments for enforcing jump conditions at each time step. When $N$ is large, the ratio of the CPU time of the MIBTD over that of the
FDTD is about 1.8. However, if a high precision is required, the MIBTD algorithm is actually more cost-efficient. For example, if one needs to achieve an accuracy level of 2.0e−3 for the case ε2 = 2.25, it can be seen in Table 2 that a dense mesh with \( N = 800 \) has to be used in the FDTD method, while a coarse mesh with \( N = 320 \) is sufficient for the MIBTD method. Referring to their CPU times in Table 3, the MIBTD method is obviously much faster.

3.2. Dispersive Case 1a: circular dispersive interface with a constant magnetic permeability

In our first dispersive case, we study a circular interface problem with magnetic permeability \( \mu \) being chosen as \( \mu_0 \) through out the domain, i.e., \( \mu^+ = \mu^- = \mu_0 \). The domain dimensions are chosen as \([0 \text{ mm}, 30 \text{ mm}]^2\), and the interface \( \Gamma \) of the Dispersive Case 1 is defined to be

\[
\Gamma : \sqrt{(x - 6\pi)^2 + (y - 15)^2} = \frac{30}{7},
\]

where the unit is also in millimeters. The circular cylinder is assumed to be a water medium with the Debye parameters:

\( \epsilon_1 = 81, \epsilon_2 = 1.8 \), and \( \gamma = 9.4 \) ps \([6, 16]\). An incident pulse of the form \( E_z(x, y, t) = \exp(-(t + \omega T)^2/(2\sigma^2)) \) is imposed as the boundary condition at the left boundary \( x = 0 \) mm. Here \( \omega = 2.5 \) mm and \( c_0 \) is the speed of light in air. The computation will stop before the propagating wave hits the right boundary \( x = 30 \) mm and the scattered wave touches the top boundary \( y = 30 \) mm and bottom boundary \( y = 0 \) mm. So, for simplicity, the incident plane waves will be imposed on the top and bottom boundaries as well, while the perfectly electric conducting (PEC) boundary condition \([29]\) is assumed on the right boundary.

The proposed second order MIBTD method is employed to solve the Maxwell–Debye system \((40)\) for this circular dispersive interface problem. For a comparison, a FDTD discretization is also considered for \((40)\). In particular, this FDTD algorithm shares the same spatial and temporal discretizations with the MIBTD method, except for the MIB interface treatment. The same boundary implementations \([30]\) are utilized in both the MIBTD and FDTD approaches. In all cases, the stop time is always assigned as \( T = 110 \) ps which is short enough such that both the propagating and scattered waves have not reached the boundaries. In the present tests, we choose the mesh size \( N \) to be odd integers. This is because that there are no analytical solutions available for the dispersive interface problems. Thus, for the purpose of benchmarking our numerical results, a reference solution is generated by employing the second order MIBTD with a very dense grid \( N = 3201 \) and a sufficiently small \( \Delta t \). In our convergence analysis, in order to make sure that the numerical values on a coarse grid are also sampled in the reference solution, the mesh size \( N \) can only be chosen as some special odd integers. In particular, the necessary requirement for a tested mesh size \( N \) is that \( N − 1 \) is an integer factor of 3200. For example, when \( N = 401 \), the reference solution needs to be downsampled with a rate 8, i.e., keeping one \( E_z \) value in every eight \( E_z \) values. The downsampled reference solution can then be compared with the numerical solution to compute the maximal error for \( E_z \). We note that since the \( H_x \) and \( H_y \) nodes are staggered to the \( E_z \) nodes, the present convergence analysis is not applicable to \( H_x \) and \( H_y \).

We first examine the stability of the dispersive MIBTD and FDTD algorithms. The CFL stability condition is also given by \((74)\). For the present dispersive interface problem with dimension units, the maximum velocity \( v_{\text{max}} \) equals to the speed of light in air, i.e., \( v_{\text{max}} = c_0 = 3 \times 10^8 \) m/s. Thus, the CFL condition can be rewritten as

\[
\Delta t = C \frac{\Delta x}{c_0 \sqrt{2}},
\]

where \( \Delta t = 110 \times 10^{-12}/(N_l - 1) \) in seconds and \( \Delta x = 30 \times 10^{-3}/(N - 1) \) in meters. Again, the value of the CFL constant \( C \) is determined by both the spatial and temporal discretizations. Similarly, we numerically detect the critical \( N_l \) value for a given \( N \) value. The critical values and corresponding CFL numbers of the MIBTD and FDTD methods are listed in Table 4. These CFL results are very similar to those of the nondispersive case given in Table 1. In particular, we also have that as \( \Delta x \) approaches to zero, the CFL number \( C \) of the FDTD method goes to \( C = \sqrt{2} \), while that of the MIBTD method is a constant within the interval \( C \in [1.21, 1.26] \). For the FDTD results, this is can be explained by the fact that the same spatial and temporal discretizations are employed in the FDTD algorithm for both the nondispersive and dispersive interface problems. For the MIBTD algorithm, there is actually a minor difference in the spatial discretization. When solving the Maxwell–Debye system \((41)\) for the nondispersive interface problem, the MIB treatment is only conducted for the \( E_z \) component, while the MIB scheme is applied to all three components in the dispersive case for solving \((40)\). Nevertheless, we note that the one-sided finite difference approximations underlying the MIB treatments for \( H_x \) and \( H_y \) are essentially the same as those.

---

**Table 4**

Critical CFL number \( C \) for the circular dispersive problem with \( \mu^- = \mu_0 \).

| \( N \) | FDTD | \( N_l \) | \( C \) | MIBTD | \( N_l \) | \( C \) |
|---|---|---|---|---|---|
| 101 | 171 | 0.9151 | 171 | 0.9151 |
| 201 | 215 | 1.4539 | 233 | 1.3411 |
| 401 | 434 | 1.4371 | 496 | 1.2571 |
| 801 | 873 | 1.4272 | 1025 | 1.2153 |
for $E_z$. More specifically, we can characterize these approximations by a parameter $L$, which is the number of grid nodes (excluding the fictitious node) being used in each jump condition discretization. It is known that the stability of the MIB scheme mainly depends on this parameter $L$ [30]. Here, we have $L = 2$ for not only $H_x$ and $H_y$, but also $E_z$. Thus, the dispersive MIBTD algorithm attains the same CFL number as the nondispersive MIBTD method.

We next study the convergence rates of the dispersive MIBTD and FDTD algorithms. To ensure the stability, we take the CFL constant to be $C = 0.7$ in both methods. The $L_\infty$ errors in $E_z$ of the dispersive FDTD and MIBTD methods are listed in Table 5. As mentioned above, the errors are calculated against the reference solution. So, the mesh size $N$ cannot be arbitrarily chosen. On the other hand, in order to obtain reasonably accurate results for the FDTD algorithm, $N$ cannot be too small. Thus, we choose $N = 321, 401, 641$ and 801 in the present study. It can be seen that with these mesh refinements, the convergence of the FDTD method is not monotonic, perhaps because that $\Delta x$ is not halved in the mesh refinements. The least squares fitting is a good tool to identify the overall numerical order in such a situation. To this end, the FDTD errors are plotted in Fig. 11. The fitted convergence rate of the FDTD method is found to be $s = 1.16$. This justifies that the FDTD method degrades to a first order method, due to the lack of proper interface treatments. However, the MIBTD method attains a second order convergence after enforcing jump conditions for $H_x$, $H_y$ and $E_z$ across the dispersive interface, as can be seen in both Table 5 and Fig. 11. The contour plots and slice plots of the MIBTD solutions are shown in Fig. 12. The slice plots obviously reveal that the $E_z$ is $C^1$ continuous across the interface but $H_x$ and $H_y$ are only $C^0$.

As in the nondispersive case, we finally investigate the speed of the dispersive MIBTD method. The CPU times in seconds consumed in the FDTD and MIBTD computations are presented in Table 6. Here, we also take the CFL constant to be $C = 0.7$ in both approaches. As one may expected, the MIBTD method spends more time than the FDTD method for the present dispersive problem, while for one method, either the MIBTD or FDTD, more CPU time is needed in the dispersive case than in the nondispersive case. We are more interested in the CPU time ratio of the MIBTD over the FDTD. It can be observed from Table 3 and Table 6 that this ratio increases from about 1.8 to about 2.3 for a large $N$ value. This is because the additional interface treatments are required in the dispersive MIBTD algorithm for $H_x$ and $H_y$. We finally note that the dispersive MIBTD method is still more cost-efficient than the dispersive FDTD method.

### 3.3. Dispersive Case 1b: circular dispersive interface with a piecewise constant magnetic permeability

In this case, the same material, domain and discretization parameters are chosen as in the previous dispersive case except the magnetic permeability value. To illustrate the effectiveness of proposed MIBTD method in handling discontinuous solutions, we set the $\mu$ value in the dispersive medium to be $2\mu_0$, i.e., $\mu^- = \mu_0$ and $\mu^+ = 2\mu_0$. It is noted that our dispersive material now becomes nonphysical.

Similar to the previous dispersive case, the numerical CFL numbers and convergence results are reported in Table 7 and Table 8 respectively. We note that the numerical CFL numbers of this case are almost identical to the ones in the previous dispersive case, even though the present MIBTD algorithm becomes more complex. This is because we also choose $L = 2$ in the MIB discretization of mixed jump conditions. The convergence rates of the MIBTD method for this case are not uniform.
Fig. 12. The MIBTD solution with $N = 321$ at $t = 110$ ps for the circular dispersive interface problem with $\mu^- = \mu_0$. (a) Contour plot of $E_z$. (b) $E_z(x, 15)$. (c) Contour plot of $H_y$. (d) $H_y(x, 15)$. (e) Contour plot of $H_x$. (f) $H_x(17.06, y)$.

Table 6

<table>
<thead>
<tr>
<th>$N$</th>
<th>CPU time of FDTD ($T_F$)</th>
<th>CPU time of MIBTD ($T_M$)</th>
<th>$T_M/T_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>1.95</td>
<td>4.31</td>
<td>2.21</td>
</tr>
<tr>
<td>161</td>
<td>8.13</td>
<td>20.80</td>
<td>2.56</td>
</tr>
<tr>
<td>201</td>
<td>15.83</td>
<td>35.94</td>
<td>2.27</td>
</tr>
<tr>
<td>321</td>
<td>67.27</td>
<td>143.50</td>
<td>2.13</td>
</tr>
<tr>
<td>401</td>
<td>141.97</td>
<td>285.67</td>
<td>2.01</td>
</tr>
<tr>
<td>641</td>
<td>551.73</td>
<td>1241.13</td>
<td>2.25</td>
</tr>
<tr>
<td>801</td>
<td>1000.19</td>
<td>2378.66</td>
<td>2.38</td>
</tr>
</tbody>
</table>

at smaller grid sizes but the second order accuracy is still achieved at larger grid sizes. Fig. 13 plots the errors of both FDTD and MIBTD methods. It can be read from it that the fitted convergence rate of the FDTD method is just $s = 0.96$. However the MIBTD method still successfully attains the second order convergence with $s = 2.14$ due to the use of advanced interface treatments.
Table 7
Critical CFL number $C$ for the circular dispersive interface problem with $\mu^- = 2\mu_0$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>FDTD Critical $N_t$</th>
<th>$C$</th>
<th>MIBTD Critical $N_t$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>101</td>
<td>171</td>
<td>0.9151</td>
<td>171</td>
<td>0.9151</td>
</tr>
<tr>
<td>201</td>
<td>215</td>
<td>1.4539</td>
<td>234</td>
<td>1.3287</td>
</tr>
<tr>
<td>401</td>
<td>434</td>
<td>1.4371</td>
<td>496</td>
<td>1.2571</td>
</tr>
<tr>
<td>801</td>
<td>874</td>
<td>1.4229</td>
<td>1021</td>
<td>1.2181</td>
</tr>
</tbody>
</table>

Table 8
Numerical convergence test of the circular dispersive interface problem in the $L_\infty$ norm with $\mu^- = 2\mu_0$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>FDTD Error</th>
<th>Order</th>
<th>MIBTD Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>321</td>
<td>9.89e−3</td>
<td>−0.27</td>
<td>1.43e−3</td>
<td>−0.28</td>
</tr>
<tr>
<td>401</td>
<td>1.05e−2</td>
<td>1.27</td>
<td>1.52e−3</td>
<td>2.62</td>
</tr>
<tr>
<td>641</td>
<td>5.79e−3</td>
<td>1.20</td>
<td>4.43e−4</td>
<td>2.62</td>
</tr>
<tr>
<td>801</td>
<td>4.44e−3</td>
<td>1.20</td>
<td>2.25e−4</td>
<td>3.04</td>
</tr>
</tbody>
</table>

Fig. 13. Numerical convergence tests for the circular dispersive interface problem with $\mu^- = 2\mu_0$. Here, the numerical errors are plotted as dashed lines, while the least-squares fitted convergence lines are shown as solid lines.

We conclude this subsection by plotting the numerical solutions of electric and magnetic fields. Fig. 14 shows that in the present case, $E_z$ is still continuous across the interface, whereas $H_x$ and $H_y$ are both discontinuous across the interface.

3.4. Dispersive Case 2: complex dispersive interfaces

In the second case of dispersive interface problems, we study arbitrarily curved interfaces with nonconstant curvatures. For simplicity, the magnetic coefficient $\mu$ will be chosen as a constant $\mu = \mu_0$ throughout the domain in the rest of this paper. A family of interfaces parameterized by the polar angle $\theta$ [24] are studied in the Dispersive Case 2

$$
\Gamma : \sqrt{(x - 6\pi)^2 + (y - 15)^2} = \frac{30}{7} + b \sin(m\theta), \quad \theta \in [0, 2\pi].
$$

The unit of this equation is in millimeters and so is that of the parameter $b$. Here the parameter $m$ determines the number of leaves of the core region and $b$ controls the magnitude of the curvature. Four independent tests with parameters $(m, b) = (2, 5/2), (3, 10/9), (4, 10/9)$ and $(5, 10/11)$ are considered as the Dispersive Case 2(a), 2(b), 2(c) and 2(d), respectively. The other problem settings including the domain dimensions, the Debye parameters, the incident wave, and boundary conditions, are the same as in the previous dispersive case. A stop time $T = 110$ ps is also employed. The results on the stability and execution time are similar to the circular dispersive interface case, and will not be presented here to save space. Essentially, the same CFL stability condition holds for the MIBTD algorithm, while the ratio of the CPU time consumed by the MIBTD over that by the FDTD is also about 2.3.

We mainly focus on the impact of the arbitrarily curved interface on the accuracy of the MIBTD method for this example. For each dispersive case, a reference solution is first generated by the MIBTD algorithm with $N = 1601$ and a sufficiently small $\Delta t$. By taking the CFL number to be $C = 0.7$, the MIBTD and FDTD solutions based on several $N$ values are compared with the reference solution on mutually sampled grid nodes. The calculated $L_\infty$ errors in $E_z$ are given in Table 9. It can be seen that the numerical order of the dispersive FDTD method is quite inconsistent. It sometimes becomes as large as 2.22, while sometimes is as low as 0.38. On the contrary, the numerical order for the MIBTD algorithm is always about 2 throughout the table. These errors are also analyzed in Fig. 15. As in the previous studies, a linear least-squares fitting is conducted in all cases. In particular, it can be seen from Figs. 16, 17, 18, and 19 that $E_z$ is $C^1$ continuous across the...
Fig. 14. The MIBTD solution with $N = 321$ at $t = 110$ ps for the circular dispersive interface problem with $\mu^- = 2\mu_0$. (a) Contour plot of $E_z$. (b) $E_z(x, 19.22)$. (c) Contour plot of $H_y$. (d) $H_y(x, 19.22)$. (e) Contour plot of $H_x$. (f) $H_x(15.66, y)$.

interface, but $H_y$ and $H_x$ are only $C^0$ continuous. Due to the loss of regularities in wave solutions near the interface, the convergence rates of the FDTD method are always found to be around first order. Specifically, the FDTD method delivers $s = 1.21$ for case 2(a), $s = 1.27$ for case 2(b), $s = 1.29$ for case 2(c), and $s = 1.30$ for case 2(d). However, by using the MIBTD method, the full second order is recovered in every case. In the present study, the numerical order $s$ of this method is found to be $s = 2.06$ for case 2(a), $s = 2.01$ for case 2(b), $s = 2.14$ for case 2(c), and $s = 2.04$ for case 2(d).

3.5. Dispersive Case 3: sharp dispersive interface

In the last case of dispersive interface problems, we study a Lipschitz continuous but not $C^1$ continuous interface. Once again, the constant magnetic permeability $\mu_0$ is employed. We define a 3/4 circle interface parameterized by an angle $\theta$ as follows

$$
\Gamma: (x, y) = \begin{cases} 
(6\pi + \frac{30}{\pi} \cos \theta, 15 + \frac{30}{\pi} \sin \theta), & \text{if } 0 \leq \theta \leq \frac{3\pi}{4} \text{ or } \frac{5\pi}{4} \leq \theta \leq 2\pi, \\
(6\pi - \frac{60\sqrt{2}}{7}(1 - \frac{\theta}{\pi}), 15 + \frac{60\sqrt{2}}{7}(1 - \frac{\theta}{\pi})), & \text{if } \frac{3\pi}{4} \leq \theta \leq \pi, \\
(6\pi + \frac{60\sqrt{2}}{7}(1 - \frac{\theta}{\pi}), 15 + \frac{60\sqrt{2}}{7}(1 - \frac{\theta}{\pi})), & \text{if } \pi \leq \theta \leq \frac{5\pi}{4}.
\end{cases}
$$

(78)
Table 9 Numerical convergence tests of the complex dispersive interface cases in the $L_{\infty}$ norm.

<table>
<thead>
<tr>
<th>N</th>
<th>Case 2(a) ($m = 2, b = 5/2$)</th>
<th>Case 2(b) ($m = 3, b = 10/9$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FDTD</td>
<td>MIBTD</td>
</tr>
<tr>
<td>101</td>
<td>1.68e−2</td>
<td>1.29e−2</td>
</tr>
<tr>
<td>161</td>
<td>1.08e−2</td>
<td>5.30e−3</td>
</tr>
<tr>
<td>201</td>
<td>7.40e−3</td>
<td>3.00e−3</td>
</tr>
<tr>
<td>321</td>
<td>4.20e−3</td>
<td>1.20e−3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>N</th>
<th>Case 2(c) ($m = 3, b = 10/9$)</th>
<th>Case 2(d) ($m = 4, b = 10/11$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FDTD</td>
<td>MIBTD</td>
</tr>
<tr>
<td>101</td>
<td>2.80e−2</td>
<td>1.20e−2</td>
</tr>
<tr>
<td>161</td>
<td>1.15e−2</td>
<td>4.30e−3</td>
</tr>
<tr>
<td>201</td>
<td>9.40e−3</td>
<td>2.70e−3</td>
</tr>
<tr>
<td>321</td>
<td>6.10e−3</td>
<td>9.80e−4</td>
</tr>
</tbody>
</table>

The unit of this equation is in millimeters. The other problem settings are the same as in the previous dispersive cases. A stop time $T = 110$ ps is also employed. The double layer MIB scheme needs to be applied to treat two sharp corners on the edge of the incomplete circle. For other irregular points, the regular MIB scheme can be used. The impact of the double layer MIB scheme on the stability and execution time of the MIBTD algorithm is negligible. In particular, it is found that the same CFL stability condition holds for the MIBTD algorithm, while the ratio of the CPU time utilized by the MIBTD over that by the FDTD is also about 2.3.

From the accuracy point of view, the double layer MIB scheme is indispensable for the present sharp dispersive interface. Without the double layer scheme, there are a few irregular nodes near the sharp corners, for which one cannot generate the necessary fictitious points for them. If neglecting these irregular nodes, the accuracy of the MIBTD algorithm would be essentially the same as that of the FDTD method. Nevertheless, after applying the double layer MIB interface treatment, the full second order can be secured throughout the domain. To illustrate this, we first generate a reference solution by the MIBTD algorithm with $N = 1601$ and a sufficiently small $\Delta t$. By taking the CFL number to be $C = 0.7$, the $L_{\infty}$ errors in
Fig. 16. The MIBTD solution with \( N = 321 \) at \( t = 110 \text{ ps} \) for the Dispersive Case 2(a) with parameters \((b, m) = (2, 5/2)\). (a) Contour plot of \( E_z \). (b) \( E_z(x, 12) \). (c) Contour plot of \( H_y \). (d) \( H_y(x, 12) \). (e) Contour plot of \( H_x \). (f) \( H_x(15, y) \).

Table 10
Numerical convergence test of the sharp dispersive interface case in the \( L_{\infty} \) norm.

<table>
<thead>
<tr>
<th>( N )</th>
<th>FDTD Error</th>
<th>FDTD Order</th>
<th>MIBTD Error</th>
<th>MIBTD Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>201</td>
<td>5.80e−3</td>
<td></td>
<td>1.33e−2</td>
<td></td>
</tr>
<tr>
<td>321</td>
<td>1.67e−2</td>
<td>−2.26</td>
<td>2.60e−3</td>
<td>3.49</td>
</tr>
<tr>
<td>401</td>
<td>5.90e−3</td>
<td>4.68</td>
<td>1.00e−3</td>
<td>4.29</td>
</tr>
<tr>
<td>801</td>
<td>1.40e−3</td>
<td>2.08</td>
<td>4.88e−4</td>
<td>1.30</td>
</tr>
</tbody>
</table>

\( E_z \) for the MIBTD and FDTD solutions are reported in Table 10 and plotted in Fig. 20. It can be seen that the order of the MIBTD is not uniform, while that of the FDTD even becomes negative in one refinement. This may be due to the presence of the corner points. In fact, it can be observed in the contour plots and slice plots shown in Fig. 21 that the solutions, particularly two magnetic components, exhibit some nearly singular behavior near two corner points. It is thus amazing to see that the overall convergence rate of the MIBTD method is 2.49 for this tough problem. This validates the effectiveness of the MIB double layer treatment and the accuracy of the entire MIBTD algorithm.
4. Conclusion

In the present study, we generalize our previous matched interface and boundary time domain (MIBTD) method [6,7] to deal with curved and/or sharp dispersive interfaces of Debye type in solving two-dimensional transverse magnetic system with both electric and magnetic coefficients being discontinuous. Based on the auxiliary differential equation approach, a hybrid Maxwell–Debye system is constructed, which couples the wave equation for the electric component with Maxwell’s equations for the magnetic components. This allows us to track the transient changes in the regularities of electromagnetic fields across the dispersive interfaces. The resulting time dependent jump conditions are rigorously enforced through regular and double layer interface treatments. Numerical results demonstrate that for both arbitrarily curved interfaces and interfaces with sharp corners, and for both continuous and discontinuous electromagnetic fields, the proposed MIBTD method can fully recover the second order of accuracy, while the classical finite difference time domain (FDTD) method yields only the first order of accuracy. The stability condition of the MIBTB method has a slightly smaller stability constant than that of the FDTD method. For a given grid, the execution time of the MIBTD method is about 2.3 times of that of the FDTD method. However, if one aims to achieve a certain accuracy level, the MIBTD method requires a much coarse grid, so that the MIBTD method is actually more cost-efficient.
In comparing with the previously developed matched interface and boundary (MIB) schemes for solving Maxwell’s equations in the time domain [6,7,23] and the Helmholtz equation in the frequency domain [24], the present MIB algorithms have three major new features:

1. In this work, a curved interface problem based on the staggered Yee lattice has been treated for the first time in the MIB literature. In the previous MIB schemes for elliptic and Helmholtz problems with curved interfaces [24], only nonstaggered meshes have been studied. Although the staggered meshes have been considered in the previous MIB schemes for Maxwell’s equations [6,7,23], only straight interface problems are involved in these studies. The straight interface greatly simplifies the numerical implementation, because the MIB interface treatment needs to be carried out only at one interface location and can be applied to other locations directly. However, the present development of MIB interface treatments for arbitrary interfaces with electric and magnetic components defined at different nodes becomes much more complicated, because the treatments at different interface locations and for different field components are all different. A sophisticated code has to be written to implement such a MIB algorithm.

2. The MIB sharp interface treatment is implemented for the first time for solving computational electromagnetic (CEM) interface problems in this work. The double layer MIB scheme was originally developed in [26] for elliptic interface...
Fig. 19. The MIBTD solution with $N = 321$ at $t = 110$ ps for the Dispersive Case 2(d) with parameters $(b, m) = (5, 10/11)$. (a) Contour plot of $E_z$. (b) $E_z(x, 11.06)$. (c) Contour plot of $H_y$. (d) $H_y(x, 11.06)$. (e) Contour plot of $H_x$. (f) $H_x(16.4, y)$.

Fig. 20. Numerical convergence tests for the sharp dispersive interface case. Here, the numerical errors are plotted as dashed lines, while the least-squares fitted convergence lines are shown as solid lines.
problems, and has never been applied to the CEM problems. With this enhancement, the proposed MIB algorithm really can handle very complicated and $C^0$ continuous interfaces.

3. The reported MIBTD method is the first MIB algorithm in the CEM literature that can handle discontinuous electromagnetic wave solutions. In our previous MIB methods [6,7,23,24], the electromagnetic solutions are all assumed to be continuous. As shown in this work, by taking $\mu^- \neq \mu^+$, the magnetic fields are discontinuous in the transverse magnetic (TM) system. For the present Maxwell–Debye problem, the jump conditions become mixed, coupled, and time dependent. A very sophisticated MIB scheme is developed in the present work to enforce such conditions to successfully secure a second order accurate FDTD simulation. This is a significant accomplishment in the development of the MIB methods for solving CEM interface problems.

It is of great interests to further generalize the proposed MIBTD method to deal with more general Maxwell systems. In particular, the development of an efficient interface scheme for solving the transverse electric (TE) system is currently under our investigation.

Fig. 21. The MIBTD solution with $N = 321$ at $t = 110$ ps for the sharp dispersive interface. (a) Contour plot of $E_z$. (b) $E_z(x, 11.72)$. (c) Contour plot of $H_y$. (d) $H_y(x, 11.72)$. (e) Contour plot of $H_x$. (f) $H_x(16.86, y)$. 
Acknowledgements

This work was supported in part by NSF grants DMS-1016579 and DMS-1318898, and the University of Alabama Research Stimulation Program (RSP) award. The authors would like to thank anonymous referees for their many insightful comments that improved the paper.

References