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Symmetric interior penalty Galerkin approaches for two-dimensional parabolic interface problems with low regularity solutions

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ABSTRACT

This work presents novel finite element approaches for solving a parabolic partial differential equation with discontinuous coefficients and low regularity solutions in a bounded convex polyhedral domain. A spatial semi-discretization based on symmetric interior penalty Galerkin (SIPG) approximations is constructed and analyzed by using discontinuous piecewise linear functions. For smooth initial data, spatial errors in the broken L^2, H^1 and $L^2(H^1)$ norms are proven to be optimal with respect to low regularity solutions, which are only piecewise H^{1+s} smooth with $0 < s \leq 1$. Furthermore we present the full SIPG discretizations based on an Euler backward finite difference time discretization. The proposed approximations are shown to be unconditionally stable, and have nearly the optimal $L^2(L^2)$ and $L^2(H^1)$ error estimates, even when the regularities of the solutions are low on the whole domain. The error estimates are optimal with respect to time, sharply depending on the indexes in the global and local regularity. Numerical experiments for two-dimensional parabolic interface problems verify the theoretical convergence rates.

1. Introduction

Parabolic interface problems appear in many physical and engineering applications, such as the continuous casting in the metallurgical industry, the freezing process of perishable foodstuffs in the food engineering, the magnetic fluid hyperthermia (MFH) treatment of cancer, as well as fluid dynamics [1–3]. Beginning with the recent nanotechnology revolution, the MFH is becoming a viable approach to targeted cancer therapy. To provide researchers a better insight in the mechanism of the MFH, Pennes's bioheat equation [4] with an appropriate heating source has been widely considered as a special case of parabolic interface problems. The analysis of conductive heat transfer process over composite media is indispensable in these applications. Here, we are mathematically interested in numerical analysis and computation of general parabolic interface problems.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with the boundary $\partial\Omega := \Gamma_D$. Suppose an interface separates the domain Ω into two subdomains $\{\Omega_i\}_{i=1}^2$ such that $\Omega_1 \subset \Omega$ with a C^2 boundary Γ, and $\Omega_2 = \Omega \setminus \Omega_1$. We denote $\mathbf{n}$ the unit normal

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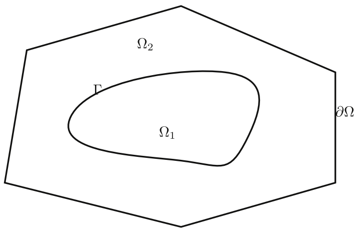

Fig. 1. Subdomains and interface.

vector to Γ. For given scalar-valued functions f, ϕ, $\tilde{\psi}$ and w_0, we consider the following linear parabolic interface problem

$$\begin{cases} \dfrac{\partial w}{\partial t} - \nabla \cdot (k(x)\nabla w) = f(x,t), & \text{in } \Omega \times (0,T], \\ w(x,0) = w_0, & \text{in } \Omega, \\ w = 0, & \text{on } \Gamma_D \times (0,T], \end{cases} \tag{1}$$

with interface conditions (see Fig. 1)

$$[w] = \phi(x,t), \quad \left[k(x)\frac{\partial w}{\partial \mathbf{n}}\right] = \tilde{\psi}(x,t) \quad \text{along } \Gamma. \tag{2}$$

Here the diffusion coefficient $k(x)$ is assumed to be positive and piecewise constant across Γ, i.e., $k(x) = k_i$ for $x \in \Omega_i$, $i = 1, 2$. We require jump and average definitions of discontinuous functions and vector functions. For any interior edge $e \in \Gamma$, there are two simplices T_+ and T_- such that $e = \partial T_+ \cap \partial T_-$. Let $\mathbf{n}_+$ be the unit normal of e pointing from T_+ to T_-, and $\mathbf{n}_- = -\mathbf{n}_+$. We define the jump and average of v on e by

$$[v] = v_+\mathbf{n}_+ + v_-\mathbf{n}_-, \quad \text{and} \quad \{v\} = \frac{1}{2}(v_+ + v_-),$$

respectively, where the notation v_+ (v_-) means the limit of $v\big|_{T_+}$ ($v\big|_{T_-}$) as T_+ (T_-) approaches to e. The interface problem (1)–(2) is often encountered in material sciences and fluid dynamics. Due to the presence of discontinuous coefficients, the solution u, in general, does not belong to $H^2(\Omega)$. Note that the most general jump conditions are considered in (2), i.e., the jump values are space and time dependent.

In order to facilitate our following theoretical analysis, with a regularization process (see Appendix A), the original problem (1) can be rewritten into

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot (k(x)\nabla u) = f(x,t), & \text{in } \Omega \times (0,T], \\ u(x,0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_D \times (0,T], \end{cases} \tag{3}$$

with interface conditions

$$[u] = 0, \quad \left[k(x)\frac{\partial u}{\partial \mathbf{n}}\right] = \psi(x,t) \quad \text{along } \Gamma, \tag{4}$$

where u_0 is defined in (A.3) and ψ is given by (A.4). The interface problem (3)–(4) is taken as a standard model to develop theoretical analysis.

For finite difference discretizations in space, the immersed interface method (IIM) was originally presented by LeVeque and Li in 1994 [5] via local Taylor's series expansions for solving elliptic interface equations. Many other elegant finite difference interface methods have been developed recently, such as maximum principle preserving and high order IIM [6,7], coupling interface method [8], and matched interface and boundary (MIB) method [9–12]. The IIM for solving parabolic equations has been investigated in [13–15]. By adding some correction terms into the classical alternating direction implicit (ADI) scheme, Liu and Zheng [16] have employed the second order accurate IIM-ADI method to solve a three-dimensional (3D) homogeneous heat equation. By avoiding the complexity associated with second order jump conditions [17], a matched ADI method has been proposed in [18] for solving linear parabolic interface problems with the most general jump conditions. The key of this method is a tensor product decomposition of the spatial and temporal dependent jump condition (2), which decouples two-dimensional (2D) conditions into one-dimensional (1D) ones.

Many finite element methods (FEMs) have been considered for solving elliptic interface problems. As for unfitted mesh methods for interface problems, Barrett and Elliott [19] showed first order of convergence in energy-norm and interior second order L^2 error estimates for a piecewise linear method. It is well known that a second order accurate approximation to solution of an interface problem can be generated by the Galerkin finite element method with the standard linear basis functions if the triangulation is aligned with the interface [20,21]. Immersed finite element (IFE) methods were studied in [22,23], in which the interface is allowed to cut through finite elements. This introduces new finite element solvers using Cartesian meshes for interface problems. Recently, a second order accurate numerical method on non-body-fitting grids has been proposed for solving the variable coefficient elliptic equation in disjoint subdomains [24]. An isoparametric type of FEM discretization is used to derive optimal order error estimates for the linear elliptic and parabolic interface problems with discontinuous coefficients (cf. [25,26]). In [27] a parabolic problem with a moving interface is discretized by a backward Euler and extended FEM method, and the corresponding errors in space and time are analyzed.

Under the framework of discontinuous Galerkin (DG) methods, a weak Galerkin finite element method (WGFEM) based on a discrete weak gradient was introduced by Wang and Ye [28] and has been successfully applied to solve elliptic interface problems in [29] recently. With the use of stabilization for the flux variable, a mixed WGFEM has been developed in [30] for the second order elliptic equation. Analogously, to weakly enforce the interface conditions, Galerkin projections and numerical fluxes have been considered in some interesting DG variational formulations [31,32]. A weighted DG method [33] as a primal form has been proposed for elliptic interface equations with discontinuous coefficients, and the sharp a priori and a posteriori error estimates have been conducted. Furthermore, the bilinear IFE functions have also been used in interior penalty discontinuous Galerkin method (IPDG) [34]. For the purpose of stabilization, interior penalty Lagrange multiplier methods [35] penalize the jump of the Lagrange multiplier over element faces.

We will develop new IPDG methods for solving interface problems, including symmetric interior penalty Galerkin (SIPG) and over-penalized symmetric interior penalty Galerkin (OPSIPG) types. Indeed, the IPDG methods inspired by the original interior penalty methods were typically presented in their primal formulation as in [36–39], permitting poorly-generated meshes and/or hanging nodes, and thus are promising in resolving geometrically complicated interfaces. Moreover, they allow a natural enforcement of the interface integrals related to the jumps of normal derivatives and weak continuity. However, in the case of fully discrete approximations, the IPDG methods have never been applied to a parabolic interface problem before. Inspired by the analysis of the fully discrete SIPG formulations for nonlinear parabolic equations in [40,41] and a post-processing gradient recovery of an OPSIPG method [42], we are particularly interested in the numerical analysis of the parabolic interface problems by the SIPG and OPSIPG schemes. Due to a low regularity of solution, numerical stability and convergence analysis is challenging for semi-discrete and fully discrete schemes approximating the parabolic interface problems, while the interface may be arbitrary smooth curves. Extensive efforts will be devoted in this work on stability and error estimates with respect to space and time for the semi-discrete and fully discrete SIPG schemes in solving parabolic interface problems with jump conditions as well as the effect of low regularities.

The rest of this paper is organized as follows. In Section 2, we first introduce the mathematical settings and the interior penalty discontinuous Galerkin formulations given in subdomains. Then the bilinear form will be estimated for its coercivity and continuity with a sharp estimation on penalty parameters. The orthogonal projection, L^2 projection and elliptic projection operators will then be estimated as some primary results in Section 3. Consequently, we will derive a priori error estimates for the semi-discrete SIPG schemes in Section 4 and for the corresponding fully discrete schemes in Section 5, respectively. Numerical validations of the proposed approximations through several examples with totally discontinuous analytical solutions are considered in Section 6, with the time-independent or time-dependent jump conditions. Finally, this paper ends with a conclusion.

2. Interior penalty discontinuous Galerkin methods

Throughout the paper, for a polygonal Lipschitz subdomain $\omega \in \Omega$, we use the standard notation and definitions for Sobolev spaces (see Adams [43]). For an open region ω, define the Sobolev space by $W^{m,p}(\omega)$ of differentiability index m and integrability index p, equipped with the standard Sobolev norm $\|\cdot\|_{W^{m,p}(\omega)}$ and seminorm $|\cdot|_{W^{m,p}(\omega)}$, where m is a real number and $1 \le p \le +\infty$. When $p = 2$, we shall write $W^{m,2}(\omega) = H^m(\omega)$ and suppress the index p in the notation of the norm and seminorm, writing $\|\cdot\|_{m,\omega}$ and $|\cdot|_{m,\omega}$, respectively. When $m = 0$, $W^{m,p}(\omega) = L^p(\omega)$. By default, $H^0(\omega)$ denotes $L^2(\omega)$ with the L^2 inner product $(\cdot,\cdot)_\omega$ and the standard L^2-norm $\|\cdot\|_{0,\omega}$, or equivalently, $\|\cdot\|_\omega$. By the trace theorem, the trace of any function in $W^{m,p}(\omega)$ lies in $W^{m-1/p,p}(\partial\omega)$, provided that $m - \frac{1}{p} > 0$ is not an integer. We shall use the Hilbert spaces $H^m_D(\omega) = \left\{ v \in H^m(\omega) : \ v = 0 \text{ on } \partial\omega \right\}$ and $H(\mathrm{div};\omega) = \left\{ \mathbf{w} \in L^2(\omega)^2 : \ \nabla\cdot\mathbf{w} \in L^2(\omega) \right\}$.

Denote the following spaces:

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \quad \text{and} \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)}$$

and

$$\|v\|_Y = \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)}.$$

Let $\mathcal{T}_h$ be a triangle subdivision of Ω into disjoint open elements $\{E\}$ such that $\bar{\Omega} = \bigcup_i \bar{E}_i$, with $h_E := diam(\bar{E})$ and $h := \max_{E\in\mathcal{T}_h} h_E$, and suppose the triangulation $\mathcal{T}_h$ along the interface has a polygonal boundary whose vertices all lie on the interface. We assume that any element $E \in \mathcal{T}_h$ has at most two vertices lying on the interface. It is assumed that the family of subdivisions $\mathcal{T}_h$ is shape regular and each element $E \in \mathcal{T}_h$ shall be an affine image of a standard reference element. Throughout this work, a positive generic constant C is assumed to be independent of h. We also use the following notation:

$\mathcal{E}_I :=$ set of all interior edges of $\mathcal{T}_h$,

$\mathcal{E}_D :=$ set of all boundary edges of $\mathcal{T}_h$.

From [21,44], it is clear that the solution u of the problem (3)–(4) belongs to the space $X \cap H_D^1(\Omega)$ and $u_t \in L^2(\Omega)$. In fact, it is mentioned in [44] that one can expect higher local regularity of the solution when the coefficients in each subdomain are more smooth locally. Thus, we assume that u_t locally satisfies that $u_t \in H^2(\Omega_i)$, $i = 1, 2$.

Recall that Green's formula:

$$\int_{\partial E} (\nabla w \cdot \mathbf{n}) v ds := \langle \nabla w \cdot \mathbf{n}, v \rangle_{\partial E} = (\Delta w, v)_E + (\nabla w, \nabla v)_E, \quad \forall E \in \mathcal{T}_h \tag{5}$$

holds for all $w \in H^{1+\epsilon}(E)$ with $\Delta w \in L^2(E)$ and for all $v \in H^{1-\epsilon}(E)$ with very small ϵ satisfying $0 < \epsilon < \frac{1}{2}$. It is well-defined that

$$\int_{\partial E} (\nabla w \cdot \mathbf{n}) v ds := \langle \nabla w \cdot \mathbf{n}, v \rangle_{\partial E},$$

where $\nabla w \cdot \mathbf{n}|_{\partial E} \in H^{\epsilon - \frac{1}{2}}(\partial E)$ and $v \in H^{\frac{1}{2}-\epsilon}(\partial E)$.

We also denote the broken Sobolev space:

$$H^r(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : v|_E \in H^r(E),\ \forall E \in \mathcal{T}_h \right\}, \tag{6}$$

and its subspace $V^r(\mathcal{T}_h) = \left\{ v \in H^r(\mathcal{T}_h) : \nabla \cdot (k\nabla) v \in L^2(E),\ \forall E \in \mathcal{T}_h \right\}$ for any real number $r > 0$. We need the following specific assumptions on $V^r(\mathcal{T}_{h,i})$, $i = 1, 2$, where $V^r(\mathcal{T}_{h,i}) := \left\{ v|_{\Omega_i} : v \in V^r(\mathcal{T}_h) \right\}$. Obviously, $V^r(\mathcal{T}_h) = V^r(\mathcal{T}_{h,1}) \cup V^r(\mathcal{T}_{h,2})$.

From a priori estimate for the solution u of the parabolic interface problem (3)–(4) in [24,28], under appropriate regularity conditions on that $u_0 \in H_D^1(\Omega)$, $f \in H^1(0, T; L^2(\Omega))$ and $\psi \in H^1(0, T; H^{1/2}(\Gamma))$, it holds that $u \in L^2(0, T; X) \cap H^1(0, T; Y)$. Since $f \in L^2(\Omega)$, then $u \in V^{1+s}(\mathcal{T}_h)$ for any $0 < s \le 1$, and the flux $\sigma = -k\nabla u \in H(\text{div}; \Omega) \cap H^s(\mathcal{T}_h)^2$. We notice that $H^2(\Omega_i) \subset V^{1+s}(\mathcal{T}_{h_i})$ for $i = 1, 2$. Here it does not include the jump conditions (4) and the homogeneous Dirichlet boundary condition, thus we shall consider them in the following weak formulation.

We introduce a variational formula for (3) defined in a proper solution space which is a subspace of piecewise $H^{1+s}(\mathcal{T}_h)$ functions for $0 < s \le 1$. Multiplying (3) by a test function $v(t) \in V^{1+s}(\mathcal{T}_h)$, integrating by parts, and using boundary conditions and the jump conditions, we have the weak formulation

$$\left(\frac{\partial u}{\partial t}, v \right) + a_\lambda(u, v) = f(v), \quad \text{for } \lambda = \{-1, 0, 1\},\ t \in (0, T], \tag{7}$$

where the bilinear form $a_\lambda(\cdot, \cdot) :\ V^{1+s}(\mathcal{T}_h) \times V^{1+s}(\mathcal{T}_h) \to \mathbb{R}$ is defined by

$$\begin{aligned} a_\lambda(u, v) = &\sum_{E \in \mathcal{T}_h} (k\nabla u, \nabla v)_E - \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k\nabla u \cdot \mathbf{n}_e\}[v] ds \\ &+ \lambda \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{k\nabla v \cdot \mathbf{n}_e\}[u] ds + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \frac{\sigma_e \{k\}}{|e|^\beta} [u][v] ds \end{aligned}$$

and a linear form is given by

$$f(v) = \sum_{E \in \mathcal{T}_h} (f, v)_E + \sum_{e \in \Gamma} \int_e \psi \{v\} ds$$

with $\beta \ge 1$ and $u(0) = u_0$. Note that the bilinear form is symmetric if $\lambda = -1$, and it is nonsymmetric otherwise. Here, the global bilinear form can be decomposed into two local weak forms

$$a_\lambda(u, v) = a_\lambda^1(u, v) + a_\lambda^2(u, v), \tag{8}$$

where the local bilinear form $a_\lambda^l(\cdot, \cdot) :\ V^{1+s}(\mathcal{T}_{h,l}) \times V^{1+s}(\mathcal{T}_{h,l}) \to \mathbb{R}$ is given by

$$a_\lambda^l(u, v) = a_\lambda(u, v)|_{\mathcal{T}_{h,l}}, \quad l = 1, 2,$$

so does $f(v)$. Analogously, we decompose that $f(v) := f^1(v) + f^2(v)$. Especially, we say that the DG method is over-penalized if $\beta > 1$; in this case, σ_e is referred to as a super-penalty parameter.

Denote the discontinuous Galerkin finite element space associated with the triangulation $\mathcal{T}_h$ by

$$\mathcal{D}_1(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) : \ v|_E \in \mathbb{P}_1(E), \ \forall E \in \mathcal{T}_h \right\},$$

where $\mathbb{P}_1(E)$ is the space of polynomials of degree 1 on element E. Now, the corresponding DG finite element method is to find $u_h \in \mathcal{D}_1(\mathcal{T}_h) \subset V^{1+s}(\mathcal{T}_h)$ such that

$$\left(\frac{\partial u_h}{\partial t}, v\right) + a_\lambda(u_h, v) = f(v), \quad \forall v \in \mathcal{D}_1(\mathcal{T}_h). \tag{9}$$

We equip the space $H^1(\mathcal{T}_h)$ with the following broken H^1-norm defined by (also referred to the energy norm)

$$\|v\|_{DG} = \left(\sum_{E\in\mathcal{T}_h} \left\| k^{1/2} \nabla v \right\|_{0,E}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \frac{\left\| \{k\}[v_h] \right\|_e^2}{|e|^\beta} \right)^{1/2}. \tag{10}$$

3. Some primary estimates

Under the same conditions, the bilinear form $a_\lambda(\cdot,\cdot)$ has the following properties:

- Coercivity:

$$a_\lambda(v_h, v_h) \geq \alpha_0 \|v_h\|_{DG}^2, \quad \forall v_h \in \mathcal{D}_1(\mathcal{T}_h), \tag{11}$$

- Continuity:

$$a_\lambda(u_h, v_h) \leq \alpha_1 \|u_h\|_{DG} \|v_h\|_{DG}, \quad \forall u_h, \ v_h \in \mathcal{D}_1(\mathcal{T}_h), \tag{12}$$

  where α_0 and α_1 are constants independent of h to be defined in the following lemmas.

In fact, we do not need the bounds of k to prove the coercivity of $a_\lambda(\cdot,\cdot)$, and also produce the optimal choices of σ_e.

Lemma 3.1. *Let $\tau = (\tau^E)_{E\in\mathcal{T}_h}$ be a vector of positive components such that τ^E (< 1) is associated to the triangle in $\mathcal{T}_h$. Assume that $\beta \geq 1$. For any edge $e \in \mathcal{E}_I$ shared by E_e^1 and E_e^2, define*

$$\sigma_e^* = \frac{9(1-\lambda)}{2} \left(\frac{\cot\theta_{E_e^1}}{\tau^{E_e^1}} + \frac{\cot\theta_{E_e^2}}{\tau^{E_e^2}} \right) |e|^{\beta-1}.$$

For any edge $e \in \mathcal{E}_D \cap \partial E_e^1$, define

$$\sigma_e^* = 18(1-\lambda) \frac{\cot\theta_{E_e^1}}{\tau^{E_e^1}} |e|^{\beta-1}.$$

Then, if $\sigma_e > \sigma_e^*$ for all $e \in \mathcal{E}_I \cup \mathcal{E}_D$, there is a constant $0 < \alpha_0(\lambda, \tau) < 1$ independent of h such that

$$a_\lambda(v_h, v_h) \geq \alpha_0 \|v_h\|_{DG}^2, \quad \forall v_h \in \mathcal{D}_1(\mathcal{T}_h),$$

where $\alpha_0(\lambda,\tau) = \min\left\{ \min_{E\in\mathcal{T}_h}(1-\tau^E), \ \min_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\left(1 - \frac{\sigma_e^*}{\sigma_e}\right) \right\}$ and $\lambda = \{-1, 0\}$. Especially, in the case $\lambda = 1$, we can choose any positive penalty parameter $\sigma_e > 0$.

Proof. Using the definition of average and Cauchy–Schwarz inequality, we have

$$\int_e \{k\nabla v_h \cdot n\}[v_h] \leq \left(\frac{k^{E_e^1}}{2} \|\nabla v_h|_{E_e^1}\|_e + \frac{k^{E_e^2}}{2} \|\nabla v_h|_{E_e^2}\|_e \right) \|[v_h]\|_e. \tag{13}$$

Analogously, for any interior edge e belonging to the boundary of element E_e^1, it holds

$$\int_e \{k\nabla v_h \cdot \mathbf{n}_e\}[v_h] ds \leq k^{E_e^1} \|\nabla v|_{E_e^1}\|_e \|[v_h]\|_e. \tag{14}$$

Recall the trace inequality on an edge of a triangle:

$$\|v_h\|_e \leq \sqrt{\frac{(p+1)(p+2)}{2} \frac{|e|}{|E|}} \|v_h\|_E,$$

where $|\cdot|$ means a measure and p is polynomials of degree on element E, written by p^E. Let θ_E be the smallest angle in a triangle E, then we derive $\frac{|e|}{|E|} \le \frac{4\cot\theta_E}{|e|}$. From the trace inequality and (13), for an interior edge $e := \partial E_e^1 \cap \partial E_e^2$, it follows

$$\begin{aligned}
&\int_e \{k\nabla v_h \cdot \mathbf{n}_e\}[v_h]ds \\
&\le \sqrt{\tau^{E_e^1}}\left\|\sqrt{k^{E_e^1}}\nabla v_h\right\|_{0,E_e^1} \frac{1}{\sqrt{\tau^{E_e^1}}}\sqrt{\frac{(p^{E_e^1}+1)(p^{E_e^1}+2)}{2}\frac{\cot\theta_{E_e^1}}{|e|}}\left\|\sqrt{k^{E_e^1}}[v_h]\right\|_e \\
&\quad + \sqrt{\tau^{E_e^2}}\left\|\sqrt{k^{E_e^2}}\nabla v_h\right\|_{0,E_e^2} \frac{1}{\sqrt{\tau^{E_e^2}}}\sqrt{\frac{(p^{E_e^2}+1)(p^{E_e^2}+2)}{2}\frac{\cot\theta_{E_e^2}}{|e|}}\left\|\sqrt{k^{E_e^2}}[v_h]\right\|_e.
\end{aligned}$$

Repeating the argument for a boundary edge $e = \mathcal{E}_D \cap \partial E_e$, we have

$$\begin{aligned}
&\int_e \{k\nabla v_h \cdot \mathbf{n}_e\}[v_h]ds \\
&\le \sqrt{\tau^{E_e}}\left\|\sqrt{k^{E_e}}\nabla v_h\right\|_{0,E_e} \frac{1}{\sqrt{\tau^{E_e^1}}}\sqrt{\frac{2(p^{E_e}+1)(p^{E_e}+2)\cot\theta_{E_e}}{|e|}}\left\|\sqrt{k^{E_e}}[v_h]\right\|_e.
\end{aligned}$$

Combining the bounds above and using Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{k\nabla v_h \cdot \mathbf{n}_e\}[v_h]ds \\
&\le \Big(3\sum_{E\in\mathcal{T}_h}\tau^E\|k^{1/2}\nabla v_h\|_E^2\Big)^{1/2}\Bigg(\sum_{e\in\mathcal{E}_I}\frac{(p^{E_e^1}+1)(p^{E_e^1}+2)}{2\tau^{E_e^1}}\frac{\cot\theta_{E_e^1}}{|e|}\left\|\sqrt{k^{E_e^1}}[v_h]\right\|_e^2 \\
&\quad + \sum_{e\in\mathcal{E}_I}\frac{(p^{E_e^2}+1)(p^{E_e^2}+2)}{2\tau^{E_e^2}}\frac{\cot\theta_{E_e^2}}{|e|}\left\|\sqrt{k^{E_e^2}}[v_h]\right\|_e^2 \\
&\quad + \sum_{e\in\mathcal{E}_D}\frac{2(p^{E_e}+1)(p^{E_e}+2)}{\tau^{E_e}}\frac{\cot\theta_{E_e}}{|e|}\left\|\sqrt{k^{E_e}}[v_h]\right\|_e^2\Bigg)^{1/2},
\end{aligned} \tag{15}$$

where $\tau^E = \max\{\tau^{E_e^1}, \tau^{E_e^2}, \tau^{E_e}\}$. By Young's inequality, it follows

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{k\nabla v_h \cdot \mathbf{n}_e\}[v_h]ds \\
&\le \sum_{E\in\mathcal{T}_h}\frac{\tau^E}{2}\|k^{1/2}\nabla v_h\|_E^2 + \sum_{e\in\mathcal{E}_I}\frac{3(p^{E_e^1}+1)(p^{E_e^1}+2)\cot\theta_{E_e^1}|e|^{\beta-1}}{4\tau^{E_e^1}}\frac{\left\|\sqrt{k^{E_e^1}}[v_h]\right\|_e^2}{|e|^\beta} \\
&\quad + \sum_{e\in\mathcal{E}_I}\frac{3(p^{E_e^2}+1)(p^{E_e^2}+2)\cot\theta_{E_e^2}|e|^{\beta-1}}{4\tau^{E_e^2}}\frac{\left\|\sqrt{k^{E_e^2}}[v_h]\right\|_e^2}{|e|^\beta} \\
&\quad + \sum_{e\in\mathcal{E}_D}\frac{3(p^{E_e}+1)(p^{E_e}+2)\cot\theta_{E_e}|e|^{\beta-1}}{\tau^{E_e}}\frac{\left\|\sqrt{k^{E_e}}[v_h]\right\|_e^2}{|e|^\beta}.
\end{aligned} \tag{16}$$

Then we get the following lower bound for $a_\lambda(v_h, v_h)$ that

$$\begin{aligned}
a_\lambda(v_h, v_h) &\ge \sum_{E\in\mathcal{T}_h}\Big(1-(1-\lambda)\frac{\tau^E}{2}\Big)\|k^{1/2}\nabla v_h\|_E^2 \\
&\quad + \sum_{e\in\mathcal{E}_I}\left(\frac{\sigma_e}{2}-(1-\lambda)\frac{3(p^{E_e^1}+1)(p^{E_e^1}+2)\cot\theta_{E_e^1}|e|^{\beta-1}}{4\tau^{E_e^1}}\right)\frac{\left\|\sqrt{k^{E_e^1}}[v_h]\right\|_e^2}{|e|^\beta}
\end{aligned} \tag{17}$$

$$+\sum_{e\in\mathcal{E}_I}\left(\frac{\sigma_e}{2}-(1-\lambda)\frac{3(p^{E_e^2}+1)(p^{E_e^2}+2)\cot\theta_{E_e^2}|e|^{\beta-1}}{4\tau^{E_e^2}}\right)\frac{\left\|\sqrt{k^{E_e^2}}[v_h]\right\|_e^2}{|e|^\beta}$$
$$+\sum_{e\in\mathcal{E}_D}\left(\sigma_e-(1-\lambda)\frac{3(p^{E_e}+1)(p^{E_e}+2)\cot\theta_{E_e}|e|^{\beta-1}}{\tau^{E_e}}\right)\frac{\left\|\sqrt{k^{E_e}}[v_h]\right\|_e^2}{|e|^\beta}.$$

Notice the fact that $\frac{1}{2}\max_{E^1\cup E^2}k(x)\le\{k\}|_e\le\max_{E^1\cup E^2}k(x)$ for $e\in\Gamma$. Thus, in the case $\lambda=1$, we can choose any positive penalty parameter $\sigma_e>0$. In the other two cases $\lambda=0$ or $\lambda=-1$, it is sufficient to prove the coercivity of a_λ under the following assumptions:

$$0<\tau^E<\frac{1}{(1-\lambda)},\quad\forall E\in\mathcal{T}_h,$$
$$\sigma_e>\frac{3(1-\lambda)}{4}\left(\frac{(p^{E_e^1}+1)(p^{E_e^1}+2)\cot\theta_{E_e^1}}{\tau^{E_e^1}}+\frac{(p^{E_e^2}+1)(p^{E_e^2}+2)\cot\theta_{E_e^2}}{\tau^{E_e^2}}\right)|e|^{\beta-1},\quad\forall e\in\mathcal{E}_I,$$
$$\sigma_e>3(1-\lambda)\frac{(p^{E_e}+1)(p^{E_e}+2)\cot\theta_{E_e}}{\tau^{E_e}}|e|^{\beta-1},\quad\forall e\in\mathcal{E}_D.$$

We conclude the proof of this lemma by taking $p=1$ on each element. □

Remark 1. In Lemma 3.1, $\sigma_e^*=\mathcal{O}\big(\cot\theta_{E_e}|e|^{\beta-1}\big)$. If θ_{E_e} is very small, then the magnitude of σ_e becomes large in the case that $\beta=1$, but it can be compromised by locally refining meshes or enlarging the value of β in the case $\beta>1$. The former case is related to the interior penalty discontinuous Galerkin (IPDG) methods, while the latter case corresponds to over-penalized interior penalty discontinuous Galerkin (OPIPDG) methods. Especially, on neighbor elements E_e^1 and E_e^2, one can choose flexible basis functions by polynomials of different approximation orders $p^{E_e^1}$ and $p^{E_e^2}$, respectively.

Analogously to Lemma 3.1, we have the following continuity lemma.

Lemma 3.2. *Let $\tau=(\tau^E)_{E\in\mathcal{T}_h}$ be a vector of positive components such that $\tau^E\,(<1)$ is associated to the triangle in $\mathcal{T}_h$. Assume that $\beta\ge1$. Then, there exists an optimal constant α_1 independent of h such that*

$$a_\lambda(w_h,v_h)\le\alpha_1\|w_h\|_{DG}\|v_h\|_{DG},\quad\forall\,w_h,v_h\in\mathcal{D}_1(\mathcal{T}_h),\tag{18}$$

where $\alpha_1=\min\Big\{\min_{E\in\mathcal{T}_h}(1+\tau^E),\ \min_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\Big(1+\frac{\sigma_e^*}{\sigma_e}\Big)\Big\}$.

From [45] and [33, Lemma 2.1], if ϵ is a positive number sufficiently small or $0<\epsilon<\frac{1}{2}$, for all $v\in H^{1+\epsilon}(E)$ with $\Delta v\in L^2(E)$, there exits a weak trace inequality that shall be used later as follows.

Lemma 3.3. *Let an edge $e\in\partial E$, $E\subset\mathcal{T}_{h,i}$ $(i=1,2)$, and very small ϵ satisfying $0<\epsilon<\frac{1}{2}$. For any $v\in H^{1+\epsilon}(E)$ with $\Delta v\in L^2(E)$, there exists a positive constant $C>0$, independent of v and h_E, such that*

$$\|\nabla v\cdot\mathbf{n}\|_{\epsilon-\frac{1}{2},e}\le C(\|v\|_{\epsilon,E}+h_E^{1-\epsilon}\|\Delta v\|_{0,E}).\tag{19}$$

Denote the conforming continuous piecewise finite element space by

$$\mathcal{C}=\{v\in H^1(\Omega):\ v|_E\in\mathbb{P}_1(E),\ \forall E\in\mathcal{T}_h\},$$

and its corresponding subspace by

$$\mathcal{C}_D=\{v\in\mathcal{C}:\ v=0,\ \text{on}\,\Gamma_D\},$$

which is the finite dimensional subspace of $H_D^1(\Omega)$ defined on $\mathcal{T}_h$ consisting of piecewise linear functions vanishing on the boundary Ω. Let $P_T:\ H_D^{1+\epsilon}(\Omega)\to\mathcal{C}_D$ be the orthogonal projection operator with respect to the inner product associated with the norm $\|k^{\frac{1}{2}}\nabla v\|_{\epsilon,\Omega}$, where the positive number ϵ is small enough satisfying $0<\epsilon<\frac{1}{2}$. Then similar to that for Proposition 2.4 in [46], the standard interpolation argument and an analysis give that for $v(t)\in H_D^{1+\epsilon}(\Omega)\cap H^{1+s}(\mathcal{T}_h)$ with $0<\epsilon\le s\le1$,

$$\left\|k^{\frac{1}{2}}\nabla(v-P_Tv)\right\|_{\epsilon,\Omega}\le C\Big(\sum_{E\in\mathcal{T}_h}h_E^{2(s-\epsilon)}\left\|k^{\frac{1}{2}}\nabla v\right\|_{s,E}\Big)^{\frac{1}{2}},\quad t\in(0,T],\tag{20}$$

where C is a positive constant independent of the mesh size and the ratio $\max_x\{k\}/\min_x\{k\}$.

Denote for any $v \in V^{1+s}(\mathcal{T}_h)$, $0 < s \leq 1$,

$$M_s(h, v) = \Big(\sum_{E\in\mathcal{T}_h} h_E^{2s}\big\|k^{\frac{1}{2}}\nabla v\big\|_{s,E}^2\Big)^{\frac{1}{2}} + \Big(\sum_{E\in\mathcal{T}_h} h_E^{2}\big\|k^{\frac{1}{2}}\Delta v\big\|_{0,E}^2\Big)^{\frac{1}{2}}.$$

It is obvious that $M_s(h, v) \leq Ch^s\|v\|_{V^{1+s}}$.

Lemma 3.4. *Assume that $0 < \epsilon \leq s \leq 1$. Let $u \in V^{1+\epsilon}(\mathcal{T}_h) \subset H^{1+s}(\mathcal{T}_h)$ and $u_h \in \mathcal{D}_1(\mathcal{T}_h)$ be the solutions of the problems* (7) and (9)*, respectively. It holds that*

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{k\nabla(P_Tu-u)\cdot\mathbf{n}_e\}[P_Tu-u_h]ds \leq CM_s(h,u)h^{\frac{\beta-1}{2}}\|P_Tu-u_h\|_{DG}. \tag{21}$$

Proof. Define by $z := P_Tu - u$, and $z_T := P_Tu - u_h$. It holds by the definition of the dual norm and the inverse inequality

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{k\nabla(P_Tu-u)\cdot\mathbf{n}_e\}[P_Tu-u_h]ds\\
&\leq \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\Big(\frac{1}{2}\|k_{E^1}\nabla z|_{E^1}\cdot\mathbf{n}_e\|_{\epsilon-\frac{1}{2},e}+\frac{1}{2}\|k_{E^2}\nabla z|_{E^2}\cdot\mathbf{n}_e\|_{\epsilon-\frac{1}{2},e}\Big)h_e^{\epsilon-\frac{1}{2}}\|[z_T]\|_{0,e}\\
&\leq \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\Big(\big\|k_{E^1}^{\frac{1}{2}}\nabla z|_{E^1}\cdot\mathbf{n}_e\big\|_{\epsilon-\frac{1}{2},e}+\big\|k_{E^2}^{\frac{1}{2}}\nabla z|_{E^2}\cdot\mathbf{n}_e\big\|_{\epsilon-\frac{1}{2},e}\Big)h_e^{\epsilon-\frac{1}{2}}\{k\}^{\frac{1}{2}}\|[z_T]\|_{0,e}.
\end{aligned}$$

Due to the trace inequality (19), it holds

$$\big\|k_{E^1}^{\frac{1}{2}}\nabla z|_{E^1}\cdot\mathbf{n}_e\big\|_{\epsilon-\frac{1}{2},e} \leq C\Big(\big\|k_{E^1}^{\frac{1}{2}}\nabla z|_{E^1}\big\|_{\epsilon,E^1}+h_{E^1}^{1-\epsilon}\big\|k_{E^1}^{\frac{1}{2}}\Delta z\big\|_{0,E^1}\Big),$$

so does the term $\big\|k_{E^2}^{\frac{1}{2}}\nabla z|_{E^2}\cdot\mathbf{n}_e\big\|_{\epsilon-\frac{1}{2},e}$. It follows from the bounds above

$$\begin{aligned}
&\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{k\nabla(P_Tu-u)\cdot\mathbf{n}_e\}[P_Tu-u_h]ds\\
&\leq C\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\Big(\sum_{\partial E\supset e}\big(\big\|k^{\frac{1}{2}}\nabla z\big\|_{\epsilon,E}+h_E^{1-\epsilon}\big\|k^{\frac{1}{2}}\Delta z\big\|_{0,E}\big)\Big)h_e^{\epsilon-\frac{1}{2}}\{k\}^{\frac{1}{2}}\|[z_T]\|_{0,e}\\
&\leq C\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\Big(\sum_{\partial E\supset e}\big(h_E^s\big\|k^{\frac{1}{2}}\nabla z\big\|_{s,E}+h_E\big\|k^{\frac{1}{2}}\Delta z\big\|_{0,E}\big)\Big)h_e^{-\frac{1}{2}}\{k\}^{\frac{1}{2}}\|[z_T]\|_{0,e}\\
&\leq Ch^{\frac{\beta-1}{2}}M_s(h,u)\|z_T\|_{DG}. \quad\square
\end{aligned}$$

Before deriving error estimates for the numerical error $e(t) := u(t) - u_h(t)$, we first define an elliptic operator $R_h : X \cap H_D^1(\Omega) \to \mathcal{D}_1(\mathcal{T}_h)$ by

$$a_\lambda(u(t) - R_hu(t), v_h) = 0, \quad \forall\, v_h \in \mathcal{D}_1(\mathcal{T}_h),\ \forall\, t \geq 0. \tag{22}$$

Approximation property of R_h are stated in the following lemma (see [47] for the proof of Lemma 3.5).

Lemma 3.5. *There exists a positive constant C independent of h such that*

$$\|v - R_hv\| + h\|v - R_hv\|_{DG} \leq Ch^2\|v\|_X, \tag{23}$$

for $v \in X \cap H_D^1(\Omega)$ and R_hv defined by (22).

Lemma 3.6. *Let u and R_hu be the solutions of* (7) and (22) *respectively with $\lambda = -1$. Assume that $u \in V^{1+s}(\mathcal{T}_h) \cap H_D^{1+\epsilon}(\Omega)$ with $0 < \epsilon < s \leq 1$ and the penalty parameters satisfy the condition $\sigma_e > \sigma_e^*$. Then*

$$\|u - R_hu\|_{DG} \leq Ch^{-\epsilon}M_s(h, u), \tag{24}$$

$$\|u - R_hu\|_{0,\mathcal{T}_h} \leq Ch^{s-2\epsilon}M_s(h, u). \tag{25}$$

Proof. Analogously to the proof of Theorem 3.2 in [33], the projection error can be split into two parts and bounded by triangle inequality:

$$\|u - R_hu\|_{DG} \leq \|u - P_Tu\|_{DG} + \|P_Tu - R_hu\|_{DG}. \tag{26}$$

Note that the projection error $u - P_T u$ is continuous and vanishes on Γ_D, thus

$$\begin{aligned}\|u - P_T u\|_{DG} &= \left\|k^{\frac{1}{2}}\nabla(u - P_T u)\right\|_{0,\Omega} \le \left\|k^{\frac{1}{2}}\nabla(u - P_T u)\right\|_{\epsilon,\Omega} \\ &\le C\Big(\sum_{E\in\mathcal{T}_h} h_E^{2(s-\epsilon)}\left\|k^{\frac{1}{2}}\nabla u\right\|_{s,E}^2\Big)^{1/2},\end{aligned} \tag{27}$$

where the last inequality comes from (20).

By the definition of R_h, the continuity of $P_T u - u$ and the coercivity of a_λ, we have

$$\begin{aligned}\alpha_0\|P_T u - R_h u\|_{DG}^2 &\le a_\lambda(P_T u - R_h u, P_T u - R_h u) = a_\lambda(P_T u - u, P_T u - R_h u) \\ &= (k\nabla(P_T u - u), \nabla(P_T u - R_h u)) \\ &\quad - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{k\nabla(P_T u - u)\cdot(\mathbf{n}_e)\}[P_T u - R_h u]ds \\ &\le C\|P_T u - u\|_{DG}\|P_T u - R_h u\|_{DG} \\ &\quad + CM_s(h,u)h^{\frac{\beta-1}{2}}\|P_T u - R_h u\|_{DG},\end{aligned}$$

where we have used Lemma 3.4 for the last inequality. The above estimate infers from Cauchy–Schwarz inequality and (27)

$$\|P_T u - R_h u\|_{DG} \le C\|P_T u - u\|_{DG} + CM_s(h,u)h^{\frac{\beta-1}{2}} \le Ch^{-\epsilon}M_s(h,u). \tag{28}$$

Combining (27) with (28) results in the first inequality (24).

To prove the second inequality (25), we consider the dual boundary value interface problem: Find $w \in V^{1+s}(\mathcal{T}_h)\cap H_D^{1+\epsilon}(\Omega)$ such that

$$a_\lambda(w, v_h) = (u - R_h u, v_h), \quad \forall\, v_h \in V^{1+s}(\mathcal{T}_h)\cap H_D^{1+\epsilon}(\Omega). \tag{29}$$

By choosing $v_h = u - R_h u$, and using the continuity of a_λ, we have

$$\begin{aligned}\|u - R_h u\|_{0,\mathcal{T}_h}^2 &= (u - R_h u, u - R_h u) = a_\lambda(w, u - R_h u) \\ &= a_\lambda(w - R_h w, u - R_h u) \\ &\le \alpha_1\|w - R_h w\|_{DG}\|u - R_h u\|_{DG}.\end{aligned} \tag{30}$$

From (24) and regularity of elliptic problems, it holds

$$\begin{aligned}\|w - R_h w\|_{DG} &\le Ch^{-\epsilon}\Big(\Big(\sum_{E\in\mathcal{T}_h} h_E^{2s}\left\|k^{\frac{1}{2}}\nabla w\right\|_{s,E}^2\Big)^{\frac{1}{2}} + \Big(\sum_{E\in\mathcal{T}_h} h_E^2 k_E\left\|\Delta w\right\|_{0,E}^2\Big)^{\frac{1}{2}}\Big) \\ &\le Ch^{s-\epsilon}\|u - R_h u\|_{0,\mathcal{T}_h}.\end{aligned}$$

Then, it follows from (30) that

$$\|u - R_h u\|_{0,\mathcal{T}_h} \le Ch^{s-\epsilon}\|u - R_h u\|_{DG} \le Ch^{s-2\epsilon}M_s(h,u).$$

This completes the proof of Lemma 3.6. □

Let $Q_h : L^2(\Omega) \to \mathcal{C}_D$ be the standard L^2 projection defined by

$$(Q_h v, \varphi) = (v, \varphi), \quad \forall\, v \in L^2(\Omega),\ \varphi \in \mathcal{C}_D, \tag{31}$$

satisfying the stability estimate

$$\|Q_h v\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega)}, \quad \forall\, v \in H_D^1(\Omega). \tag{32}$$

Especially, if we replace Ω by $\mathcal{T}_h$ in (32), it still holds (see [48, Lemma 3.3]).

Lemma 3.7. *Set $\lambda = -1$. Let R_h and Q_h be defined by* (22) and (31), *respectively. Then we have*

$$\|Q_h v - v\|_{H^1(\mathcal{T}_h)} \le C\|R_h v - v\|_{DG}, \quad \forall\, v \in H_D^1(\Omega)\cap X. \tag{33}$$

Proof. For any $v \in H_D^1(\Omega)\cap X$, by the coercivity and continuity of the bilinear form $a_\lambda(\cdot,\cdot)$, there exists a unique solution $\chi \in X\cap H_D^1(\Omega)$ for the elliptic interface problem

$$a_\lambda(\chi, \varphi) = (R_h v - v, \varphi), \quad \forall\, \varphi \in X\cap H_D^1(\Omega). \tag{34}$$

Eq. (34), together with (22), and Lemma 3.5 lead to

$$\begin{aligned}\|R_h v - v\|^2_{L^2(\Omega)} &= a_\lambda(\chi - R_h\chi, R_h v - v) + a_\lambda(R_h\chi, R_h v - v)\\ &\le C\|\chi - R_h\chi\|_{DG}\|R_h v - v\|_{DG}\\ &\le Ch\|\chi\|_X\|R_h v - v\|_{DG}\\ &\le Ch\|R_h v - v\|_{L^2(\Omega)}\|R_h v - v\|_{DG},\end{aligned}\tag{35}$$

where we have used the fact that $\|\chi\|_X \le C\|R_h v - v\|_{L^2(\Omega)}$ for the last inequality. Using the triangle inequality and the inverse inequality yields

$$\begin{aligned}\|Q_h v - v\|_{H^1(\mathcal{T}_h)} &\le \|R_h v - v\|_{H^1(\mathcal{T}_h)} + \|R_h v - Q_h v\|_{H^1(\mathcal{T}_h)}\\ &= \|R_h v - v\|_{H^1(\mathcal{T}_h)} + \sum_{E\in\mathcal{T}_h}\|R_h v - Q_h v\|_{H^1(E)}\\ &\le \|R_h v - v\|_{H^1(\mathcal{T}_h)} + Ch^{-1}\sum_{E\in\mathcal{T}_h}\|Q_h v - R_h v\|_{L^2(E)}\\ &\le \|R_h v - v\|_{H^1(\mathcal{T}_h)} + Ch^{-1}\sum_{E\in\mathcal{T}_h}\big(\|v - R_h v\|_{L^2(E)} + \|v - Q_h v\|_{L^2(E)}\big).\end{aligned}\tag{36}$$

It is well known that $Q_h v$ is the best approximation of $v \in L^2(\Omega)$ with respect to the L^2 norm. Thanks to $v \in H^1_D(\Omega)\cap X$ and $R_h v \in \mathcal{D}_1(\mathcal{T}_h)$, it holds

$$\|v - Q_h v\|_{L^2(\mathcal{T}_h)} \le C\|v - R_h v\|_{L^2(\mathcal{T}_h)}.$$

Consequently, (36) implies

$$\|Q_h v - v\|_{H^1(\mathcal{T}_h)} \le \|R_h v - v\|_{H^1(\mathcal{T}_h)} + Ch^{-1}\|v - R_h v\|_{L^2(\mathcal{T}_h)}.\tag{37}$$

We conclude from (35) and (37). □

Lemma 3.8. *Let u be the solution for the parabolic interface problem (3)–(4) with $\lambda = -1$ and Q_h be defined by (31). Assume that $u \in V^{1+s}(\mathcal{T}_h)\cap H^{1+\epsilon}_D(\Omega)$ with $0 < \epsilon < s \le 1$. Then, we have*

$$\sum_{i=1}^{2}\|Q_h u_t - u_t\|_{H^m(\mathcal{T}_{h,i})} \le Ch^{(2-m)(s-\epsilon)}\big(\|u_t\|_{V^{1+s}(\mathcal{T}_{h_1})} + \|u_t\|_{V^{1+s}(\mathcal{T}_{h_2})}\big),\quad m = 0, 1,\tag{38}$$

where the constant C is independent of h.

Proof. It is easy to verify that $u_t = 0$ on $\partial\Omega$, then Lemma 3.6 leads to

$$\|u_t - R_h u_t\|_{DG} \le Ch^{-\epsilon}M_s(h, u_t) \le Ch^{s-\epsilon}\big(\|u_t\|_{V^{1+s}(\mathcal{T}_{h_1})} + \|u_t\|_{V^{1+s}(\mathcal{T}_{h_2})}\big),\tag{39}$$

$$\|u_t - R_h u_t\|_{0,\mathcal{T}_h} \le Ch^{s-2\epsilon}M_s(h, u_t) \le Ch^{2s-2\epsilon}\big(\|u_t\|_{V^{1+s}(\mathcal{T}_{h_1})} + \|u_t\|_{V^{1+s}(\mathcal{T}_{h_2})}\big).\tag{40}$$

The proof for L^2 norm estimate follows from the fact that

$$\sum_{i=1}^{2}\|Q_h u_t - u_t\|_{0,\mathcal{T}_{h,i}} \le C\|u_t - R_h u_t\|_{0,\mathcal{T}_h}.$$

Arguing as Lemma 3.7 in each subdomain and summing them, we have

$$\sum_{i=1}^{2}\|Q_h u_t - u_t\|_{H^1(\mathcal{T}_{h,i})} \le C\|u_t - R_h u_t\|_{DG}.$$

Consequently, the desired result follows from (39)–(40). □

4. A priori error estimates for the semi-discrete SIPG schemes

Theorem 4.1. *Let u and u_h be the solutions of the problem* (7) and (9). *Assume that $u \in V^{1+s}(\mathcal{T}_h)\cap H^{1+\epsilon}_D(\Omega)$ and $u_t \in L^2(0, T; V^{1+s}(\mathcal{T}_h))$ with $0 < \epsilon < s \le 1$. Then, for $u_0 \in H^1_D(\Omega)$, $f \in H^1(0, T; L^2(\Omega))$, and $\psi \in H^1(0, T; H^{\frac{1}{2}}(\Gamma))$, there exists a* *positive constant C independent of h such that*

$$\|u(t)-u_h(t)\|_{L^2(\mathcal{T}_h)} \leq Ch^{(2s-2\epsilon)}\Big(\big\|u(t)\big\|_{V^{1+s}(\mathcal{T}_h)}+\|u_t\|_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,1}))}+\|u_t\|_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,2}))}\Big).$$

Proof. Due to the linearity of a_λ, and (22), one can see that

$$a_\lambda\Big(\frac{d}{dt}(u-R_hu)(t),v_h\Big)=0,\quad \forall\, v_h\in\mathcal{D}_1(\mathcal{T}_h),\ \forall\, t\geq 0,$$

which means the time derivation and the elliptic projection are commutative.

Since the weak formulations are consistent, the following orthogonality equation holds:

$$\Big(\frac{\partial(u-u_h)}{\partial t},v_h\Big)+a_\lambda(u(t)-u_h(t),v_h)=0,\quad \forall\, v_h\in\mathcal{D}_1(\mathcal{T}_h),\ \forall\, t\geq 0. \tag{41}$$

Define $\eta=u_h-R_hu$. Then we have from (41) and the definition of the elliptic projection

$$\begin{aligned}\Big(\frac{\partial\eta(t)}{\partial t},v_h\Big)+a_\lambda(\eta(t),v_h)&=\Big(\frac{\partial u(t)-R_hu(t)}{\partial t},v_h\Big)+a_\lambda(u(t)-R_hu(t),v_h)\\&=\Big(\frac{\partial u(t)-R_hu(t)}{\partial t},v_h\Big).\end{aligned}$$

Taking $v_h=\eta(t)$ and applying the coercivity of a_λ leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{0,\mathcal{T}_h}+\alpha_0\|\eta\|^2_{DG}&\leq\Big|\Big(\frac{\partial(u(t)-R_hu(t))}{\partial t},\eta\Big)\Big|\\&\leq\frac{\alpha_0}{2}\|\eta\|^2_{DG}+\frac{1}{2\alpha_0}\Big\|\frac{\partial(u(t)-R_hu(t))}{\partial t}\Big\|^2_{0,\mathcal{T}_h}.\end{aligned}$$

With the use of the error estimate (25) satisfied by the elliptic projection and the fact that the elliptic projection and its time derivation are commutative, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{0,\mathcal{T}_h}+\frac{\alpha_0}{2}\|\eta\|^2_{DG}&\leq\frac{C}{2\alpha_0}h^{(2s-4\epsilon)}M_s(h,u_t)^2\\&\leq Ch^{(4s-4\epsilon)}\big(\|u_t\|^2_{V^{1+s}(\mathcal{T}_{h,1})}+\|u_t\|^2_{V^{1+s}(\mathcal{T}_{h,2})}\big).\end{aligned}$$

Integrating the inequality above from 0 to t and by noting that $\eta(0)=0$ yields

$$\|\eta(t)\|^2_{0,\mathcal{T}_h}+\int_0^t\|\eta\|^2_{DG}ds\leq Ch^{(4s-4\epsilon)}\big(\|u_t\|^2_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,1}))}+\|u_t\|^2_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,2}))}\big),$$

where the constant C only depends on $k_{\max}$ and α_0, independent of h. By using the triangle inequalities in the L^2 norm

$$\|u(t)-u_h(t)\|_{0,\mathcal{T}_h}\leq\|\eta(t)\|_{0,\mathcal{T}_h}+\|u(t)-R_hu(t)\|_{0,\mathcal{T}_h},$$

we have the following error estimates in the pointwise-in-time (L^2) norm

$$\|u(t)-u_h(t)\|_{L^2(\mathcal{T}_h)} \leq Ch^{(2s-2\epsilon)}\Big(\big\|u(t)\big\|_{V^{1+s}(\mathcal{T}_h)}+\|u_t\|_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,1}))}+\|u_t\|_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,2}))}\Big).\quad\square$$

Theorem 4.2. *Let u and u_h be the solutions of the problem* (7) and (9). *Assume that $u\in V^{1+s}(\mathcal{T}_h)\cap H^{1+\epsilon}_D(\Omega)$ and $u_t\in L^2(0,T;V^{1+s}(\mathcal{T}_h))$ with $0<\epsilon<s\leq 1$. Then, for $u_0\in X\cap H^1_D(\Omega)$, $f\in H^1(0,T;L^2(\Omega))$, and $\psi\in H^1(0,T;H^{\frac{1}{2}}(\Gamma))$, there exists a positive constant C independent of h such that*

$$\int_0^t\|e\|^2_{DG}ds\leq Ch^{2s-2\epsilon}\left(\|u_0\|^2_{V^{1+s}(\mathcal{T}_h)}+h^{2-2s+2\epsilon}\|u\|^2_{H^1(\Omega)}+\sum_{i=1}^{2}\|u\|^2_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,i}))}+\sum_{i=1}^{2}h^{2(s-\epsilon)}\|u_t\|^2_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,i}))}\right),$$

and

$$\|e\|_{DG}^2 \le Ch^{2s-2\epsilon}\left(\|u_0\|_{V^{1+s}(\mathcal{T}_h)}^2 + h^{2-2s+2\epsilon}\|u\|_{H^1(\Omega)}^2 + \sum_{i=1}^{2}\|u\|_{L^2(0,T;H^2(\Omega_i))}^2 + \sum_{i=1}^{2}\|u_t\|_{L^2(0,T;V^{1+s}(\mathcal{T}_{h,i}))}^2\right).$$

Proof. Since the weak formulations are consistent, the following orthogonality equation holds:

$$\left(\frac{\partial(u-u_h)}{\partial t}, v_h\right) + a_\lambda(u(t)-u_h(t), v_h) = 0, \quad \forall\, v_h \in \mathcal{D}_1(\mathcal{T}_h),\ \forall\, t \ge 0. \tag{42}$$

Define the error $e(t)$ by $e(t) = u(t) - u_h(t)$, and by its time derivative $e_t := \frac{\partial e}{\partial t} = u_t - u_{ht}$. Taking $v_h = Q_h u_t - u_{ht}$, we have

$$\begin{aligned}
&(e_t, e_t) + a_\lambda(e, e_t)\\
&= a_\lambda(e, u_t - Q_h u_t) + (e_t, u_t - Q_h u_t)\\
&= a_\lambda(e, u_t - Q_h u_t) + (u_t - Q_h u_t, u_t - Q_h u_t) + (Q_h u_t - u_{ht}, u_t - Q_h u_t)\\
&= a_\lambda(e, u_t - Q_h u_t) + (u_t - Q_h u_t, u_t - Q_h u_t) + (Q_h u_{ht} - u_{ht}, u_t - Q_h u_t),
\end{aligned}$$

where in the last equality we have used the fact that $Q_h u_t - Q_h u_{ht} \in \mathcal{C}_D$ and the definition of operator Q_h. Due to the fact that $\|u_t - Q_h u_t\|_{L^2(\Omega)} \le \|u_t - u_{ht}\|_{L^2(\Omega)} = \|e_t\|_{L^2(\Omega)}$. Then, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}a_\lambda(e, e) &\le \alpha_1\|e\|_{DG}\|u_t - Q_h u_t\|_{H^1(\Omega)} + \|Q_h u_t - u_{ht}\|_{0,\mathcal{T}_h}\|u_t - Q_h u_t\|_{0,\mathcal{T}_h}\\
&\le C\Big(\|e\|_{DG}^2 + \|u_t - Q_h u_t\|_{H^1(\Omega)}^2\Big).
\end{aligned}$$

Integrating the above equation from 0 to t leads to

$$a_\lambda(e, e) \le a_\lambda(e(0), e(0)) + C\left(\int_0^t \|e\|_{DG}^2 ds + \int_0^t \|u_t - Q_h u_t\|_{H^1(\Omega)}^2 ds\right),$$

which gives rise to

$$\alpha_0\|e\|_{DG}^2 \le \alpha_1\|e(0)\|_{DG}^2 + C\left(\int_0^t \|e\|_{DG}^2 ds + \int_0^t \|u_t - Q_h u_t\|_{H^1(\Omega)}^2 ds\right) \tag{43}$$

because of the coercivity of the bilinear form. Again, setting $v_h = Q_h u - u_u$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|e\|_{0,\mathcal{T}_h}^2 + a_\lambda(e, e)\\
&= a_\lambda(e, u - Q_h u) + (u_t - u_{ht}, u - Q_h u)\\
&= a_\lambda(e, u_t - Q_h u) + (u_t - Q_h u_t, u - Q_h u) + (Q_h u_t - u_{ht}, u - Q_h u)\\
&= a_\lambda(e, u - Q_h u) + \frac{1}{2}\frac{d}{dt}(u - Q_h u, u - Q_h u) + (Q_h u_{ht} - u_{ht}, u - Q_h u),
\end{aligned} \tag{44}$$

where

$$(Q_h u_{ht} - u_{ht}, u - Q_h u) = \frac{d}{dt}(Q_h u_h - u_h, u - Q_h u) - (Q_h u_h - u_h, u_t - Q_h u_t).$$

Thanks to $\|u_h - Q_h u_h\|_{L^2(\Omega)}^2 \le \|u_h - \mathcal{O}_s u_h\| \le Ch^{1/2}\big(\sum_{e\in\mathcal{E}_I}\|[u_h]\|_{0,e}^2\big)^{1/2} \le Ch^{1/2}\|e\|_{DG}$, integrating (44) from 0 to t gives

$$\int_0^t \|e\|_{DG}^2 ds \le C\left(\|e(0)\|_{DG}^2 + \|u - Q_h u\|_{L^2(\Omega)}^2 + \int_0^t \big(\|u - Q_h u\|_{H^1(\Omega)}^2 + \|u_t - Q_h u_t\|_{L^2(\Omega)}^2\big)ds\right).$$

An application of Lemma 3.7 gives

$$\int_0^t \|e\|_{DG}^2 ds \le C\left(\|e(0)\|_{DG}^2 + \|u - Q_h u\|_{L^2(\Omega)}^2 + \int_0^t \big(\|u - R_h u\|_{DG}^2 + \|u_t - Q_h u_t\|_{L^2(\Omega)}^2\big)ds\right).$$

Using the above inequality together with (24) and Lemma 3.8, we have

$$\int_0^t \|e\|_{DG}^2 ds \le C\left(\|e(0)\|_{DG}^2 + h^2\|u\|_{H^1(\Omega)}^2 + h^{-2\epsilon}\int_0^t M_s(h,u)^2 ds + h^{4(s-\epsilon)}\sum_{i=1}^2\int_0^t \|u_t\|_{V^{1+s}(\mathcal{T}_h)}^2 ds\right). \tag{45}$$

Finally, the proof is concluded by $\|e(0)\|_{DG} = \|u_0 - R_h u_0\|_{DG} \le Ch^{-\epsilon}M_s(h,u_0) \le Ch^{s-\epsilon}\|u_0\|_{V^{1+s}(\mathcal{T}_h)}$, and inserting (45) into (43). □

5. A priori error estimates and numerical stability for fully discrete schemes

We first divide the time interval $(0,T)$ into N equally-spaced subintervals by the following points

$$0 = t_0 < t_1 < \cdots < t_N = T$$

with $t_n = n\triangle t$ and $\triangle t = T/N$ the time step size.

Let $I_n = (t_{n-1}, t_n]$ be the nth sub-interval. For any sequence $\{\phi^n\}_{n=0}^N \subset L^2(\mathcal{T}_h)$, we introduce the backward difference quotient

$$\widetilde{\nabla}\phi^n = \frac{\phi^n - \phi^{n-1}}{\triangle t}.$$

We use the finite-dimensional space

$$V_{hk} = \{\phi : [0,T] \to \mathcal{D}_1(\mathcal{T}_h) : \phi|_{I_n} \in \mathcal{D}_1(\mathcal{T}_h) \text{ is constant in time}\}.$$

For $\phi \in \mathcal{D}_1(\mathcal{T}_h)$, we let ϕ^n be the value of ϕ at t_n. Furthermore, we write V_{hk}^n for the restriction to I_n of the functions in V_{hk}. Set $\psi_h \in \mathcal{C}_D$ by

$$\psi_h = \sum_{j=1}^{m_h} \psi(P_j)\phi_j^h$$

where $\{\phi_j^h\}_{j=1}^{m_h}$ is the set of standard nodal basis functions corresponding to the nodes $\{P_j\}_{j=1}^{m_h}$ on the interface Γ. For a given Banach space $\mathcal{B}$ and a function $\xi \in L^2(0,T;\mathcal{B})$, we write

$$\bar{\xi}^n(\cdot) = \frac{1}{\triangle t}\int_{I_n} \xi(\cdot,t)dt. \tag{46}$$

The fully discrete DG finite element approximation to (9) is defined as follows. For each $n = 1,2,\ldots,N$, find $u_h^n \in V_{hk}^n$ such that

$$(\widetilde{\nabla}u_h^n, v_h) + a_\lambda(u_h^n, v_h) = (f^n, v_h) + (\bar{\psi}_h^n, v_h)_{\Gamma_h}, \quad \forall\, v_h \in V_{hk}^n, \tag{47}$$

with $u_h^0 = Q_h u_0$. Here $\bar{\psi}_h^n = \sum_{j=1}^{m_h}\bar{\psi}^n(P_j)\phi_j^h$ with $\bar{\psi}^n$ defined as in (46). Evidently, by Lax–Milgram theorem, problem (47) has a unique solution $u_h^n \in V_{hk}^n$, for $n = 1,2,\ldots,N$.

The next lemma describes the accuracy between the interface function $\psi(x,t)$ and its discrete approximation ψ_h (see [21, Lemma 2.2]).

Lemma 5.1. *Let $\psi \in H^2(\Gamma)$. If Ω^* is the union of all interface triangles, then*

$$\left|\int_\Gamma \psi v_h ds - \int_{\Gamma_h}\psi_h v_h ds\right| \le Ch^{\frac{3}{2}}\|\psi\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega^*)}, \quad \forall\, v_h \in \mathcal{C}_D. \tag{48}$$

Furthermore,

$$\left|\int_\Gamma \psi v_h ds - \int_{\Gamma_h}\psi_h v_h ds\right| \le Ch^{\frac{3}{2}}\|\psi\|_{H^2(\Gamma)}\|v_h\|_{DG,\Omega^*}, \quad \forall\, v_h \in \mathcal{D}_1(\mathcal{T}_h), \tag{49}$$

where $\|v_h\|_{DG,\Omega^*}$ means the broken energy norm $\|v_h\|_{DG}$ of v_h defined in Ω^*.

Then, we prove the following stability result for the solution u_h^n satisfying (47).

Lemma 5.2. *Let u_h^n be the solution of (47) and $\psi \in H^2(\Gamma)$. Then it holds*

$$\|u_h^N\|_{L^2(\mathcal{T}_h)}^2 + \sum_{n=1}^N \triangle t\|u_h^n\|_{DG}^2 \le C\Big(\|u_0\|_{L^2(\Omega)}^2 + \|f\|_{L^2(0,T;L^2(\Omega))} + \|\psi\|_{L^2(0,T;H^2(\Gamma))}^2\Big). \tag{50}$$

Proof. Setting $v_h = u_h^n$ in (47) and then using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\left\|u_h^n\right\|_{0,\mathcal{T}_h}^2 - \left\|u_h^{n-1}\right\|_{0,\mathcal{T}_h}^2 + \left\|u_h^n - u_h^{n-1}\right\|_{0,\mathcal{T}_h}^2 + 2\alpha_0 \triangle t \|u_h^n\|_{DG}^2 \\
&\leq \triangle t(f^n, u_h^n) + \triangle t(\bar{\psi}_h^n, \{u_h^n\})_{\Gamma_h} - \triangle t(\bar{\psi}^n, \{u_h^n\})_{\Gamma} + \triangle t(\bar{\psi}^n, \{u_h^n\})_{\Gamma} \\
&\leq \triangle t\|f^n\|_{0,\mathcal{T}_h}\left\|u_h^n\right\|_{0,\mathcal{T}_h} + C\triangle t\|\bar{\psi}^n\|_{H^2(\Gamma)}\|u_h^n\|_{DG},
\end{aligned}$$

where we have used Lemma 5.1 and Poincaré inequality in the last inequality.

Summing over n from $n = 1$ to $n = N$, and using the discrete Gronwall's inequality, we have

$$\begin{aligned}
&\left\|u_h^N\right\|_{0,\mathcal{T}_h}^2 + \sum_{n=1}^{N} \triangle t\|u_h^n\|_{DG}^2 \\
&\leq C\Big(\left\|u^0\right\|_{0,\Omega}^2 + \sum_{n=1}^{N} \triangle t\|f^n\|_{0,\Omega}^2 + \sum_{n=1}^{N} \triangle t\|\bar{\psi}^n\|_{H^2(\Gamma)}^2\Big).
\end{aligned} \tag{51}$$

Notice the fact that

$$\sum_{n=1}^{N} \triangle t\|f^n\|_{0,\Omega}^2 \leq C\|f\|_{L^2(0,T;L^2(\Omega))}^2, \tag{52}$$

$$\sum_{n=1}^{N} \triangle t\|\bar{\psi}^n\|_{H^2(\Gamma)}^2 \leq C\|\psi\|_{L^2(0,T;H^2(\Gamma))}^2. \tag{53}$$

Combining the estimate (51) with (52)–(53) leads to the conclusion. □

Next, we state the main results of this section in the following theorems.

Theorem 5.3. *Let u and u_h^n be the solutions of* (9) and (47)*, respectively. Let $\lambda = -1$. For $f \in H^1(0, T; L^2(\Omega))$, $\psi \in L^2(0, T; H^2(\Gamma))$ and $u_0 \in X \cap H_D^1(\Omega)$, there exists a positive constant C independent of h and $\triangle t$ such that*

$$\begin{aligned}
&\Big(\int_0^T \left\|u - u_h^n\right\|_{0,\mathcal{T}_h}^2 dt\Big)^{\frac{1}{2}} \\
&\leq M_1^*(u, \psi, f)\Big(\Delta t + h^{2(s-\epsilon)} + h^{\frac{3}{2}} \max\big\{h^{\frac{1}{2}}|\log(h)|^{\frac{1}{2}}, h^{s-\epsilon}\big\}\Big),
\end{aligned} \tag{54}$$

where

$$\begin{aligned}
M_1^*(u, \psi, f) &:= C\|u_0\|_X \\
&+ C\Big(\int_0^T \big(\|u\|_X^2 + \|u_t\|_{L^2(\Omega)}^2 + \|u\|_{V^{1+s}(\mathcal{T}_h)}^2 + \|f_t\|_{L^2(\Omega)}^2 + \|\psi\|_{H^2(\Gamma)}^2\big)dt\Big)^{\frac{1}{2}}.
\end{aligned}$$

Its proof is given in Appendix B.

The following theorem states the energy-norm error estimate, which is proved in Appendix C.

Theorem 5.4. *Let u and u_h^n be the solutions of* (9) and (47)*, respectively. Let $\lambda = -1$. For $f \in H^1(0, T; L^2(\Omega))$, $\psi \in L^2(0, T; H^2(\Gamma))$ and $u_0 \in X \cap H_D^1(\Omega)$, there exists a positive constant C independent of h and $\triangle t$ such that*

$$\Big(\int_0^T \left\|u - u_h^n\right\|_{DG}^2 dt\Big)^{\frac{1}{2}} \leq M_2^*(u, \psi, f)\big(\Delta t + h^{s-\epsilon}\big), \tag{55}$$

where

$$\begin{aligned}
M_2^*(u, \psi, f) &:= C\|u_0\|_{V^{1+s}(\mathcal{T}_h)} \\
&+ C\Big(\int_0^T \big(\|u\|_{V^{1+s}(\mathcal{T}_h)}^2 + \|u_t\|_Y^2 + \|f_t\|_{L^2(\Omega)}^2 + \|\psi\|_{H^2(\Gamma)}^2\big)dt\Big)^{\frac{1}{2}}.
\end{aligned}$$

6. Numerical tests

In this section, we investigate stability and accuracy of the fully discrete SIPG scheme ($\lambda = -1$) for solving 2D parabolic interface problems with different jump conditions and interface geometries. The penalty parameter $\sigma_e = 0.25$ and the index $\beta = 3$ are taken in (47) as an OPSIPG scheme to apply in the following examples.

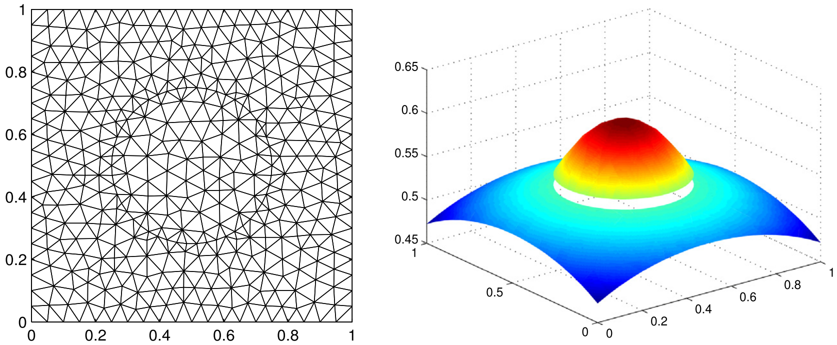

Fig. 2. Plots of Example 1. Left: A Delaunay mesh matched with the interface; Right: numerical solution ($T = 1$).

Table 1
Convergence rates in time for errors in the L^2 and L^∞ norms.

Time steps	$\|error\|_{L^2(H^1)}$	*order*	$\|error\|_{L^2(L^2)}$	*order*
$\triangle t$	7.3316E−02	–	7.3049E−02	–
$\triangle t/2$	3.8502E−02	0.929	3.7907E−02	0.946
$\triangle t/4$	2.0538E−02	0.907	1.9315E−02	0.973
$\triangle t/8$	1.2078E−02	0.766	9.7685E−03	0.984

Table 2
Convergence rates in space.

Mesh sizes	$\|error\|_{L^2(H^1)}$	*order*	$\|error\|_{L^2(L^2)}$	*order*
h	7.2224E−03	–	1.8248E−04	–
$h/2$	2.4782E−03	1.543	4.2007E−05	2.119
$h/4$	1.0382E−03	1.255	1.9955E−05	1.074

Example 1. We consider a circular interface problem with constant jump values. A square domain $[0, 1] \times [0, 1]$ is defined with a circular interface $(x - 0.5)^2 + (y - 0.5)^2 = r_0^2 := 0.25^2$ on Γ. The coefficient k is defined to be $k_1 = 1$ and $k_2 = b$, respectively on each subdomain, for $r_0 < 0.25$ and $r_0 \geq 0.25$. The analytical solution to the parabolic equation is given as

$$w(x, y, t) = \begin{cases} -((x-0.5)^2 + (y-0.5)^2 - 0.25^2) + \cos(t), & r \leq 0.25, \\ -(0.25^2(1 - \dfrac{1}{8b} - \dfrac{1}{b}) + (\dfrac{r_0^4}{2} + r_0^2))/10 + \cos(t), & \text{otherwise.} \end{cases}$$

By choosing $b = 10$, it can be checked that on the interface $[w] = -\frac{307}{25600}$, $\left[k(x)\frac{\partial w}{\partial \mathbf{n}}\right] = \frac{143}{320}$. Here the final time $T = 1$ is taken. A Delaunay mesh fitted with the circular interface (Fig. 2 Left) has been used for simulation. Accordingly, Fig. 2 Right displays numerical solutions in 3D with a constant jump along the interface.

The relative errors in the $L^2(L^2)$ and $L^2(H^1)$ norms are defined by the forms

$$\|error\|^2_{L^2(L^2)} := \sum_{n=1}^{N} \triangle t\left(\left\|u_h^n\right\|^2_{0,\mathcal{T}_h} / \left\|u^n\right\|^2_{0,\mathcal{T}_h}\right),$$
$$\|error\|^2_{L^2(H^1)} := \sum_{n=1}^{N} \triangle t\left(\|u^n - u_h^n\|^2_{DG} / \|u^n\|^2_{H^1}\right).$$

To test the temporal order, we employ SIPG scheme with piecewise linear elements on a quasi-uniform mesh of size $h = 5.0965E{-}02$ matched with the circular interface and take the initial time step $\triangle t = 1E{-}1$ till the final time $T = 1$. Table 1 shows a first-order convergence rate in the $L^2(L^2)$ norm, which appears to be better than the convergence rates in the $L^2(H^1)$ norm. To get spatial convergence rates, we consider piecewise linear element on an initial mesh of size $h = 5.0965E{-}02$ matched with the circular interface, and a time step $\triangle t = 2E{-}5$. In Table 2, our method achieves second order convergence rates in the $L^2(H^1)$ norm, and convergence rates are about 1.6 in the $L^2(L^2)$ norm.

Example 2. We next study an interface with general shape. Consider a heat equation defined in a square domain $[-1, 1]^2$ with discontinuous diffusion coefficients k_i in subdomains Ω_i $(i = 1, 2)$, where piecewise constants $k_1 = 1$ and $k_2 = 10$ are taken. An interface parameterized with the polar angle θ is defined by

$$\Gamma : r = \frac{1}{2} + b\sin(m\theta), \quad \theta \in [0, 2\pi], \tag{56}$$

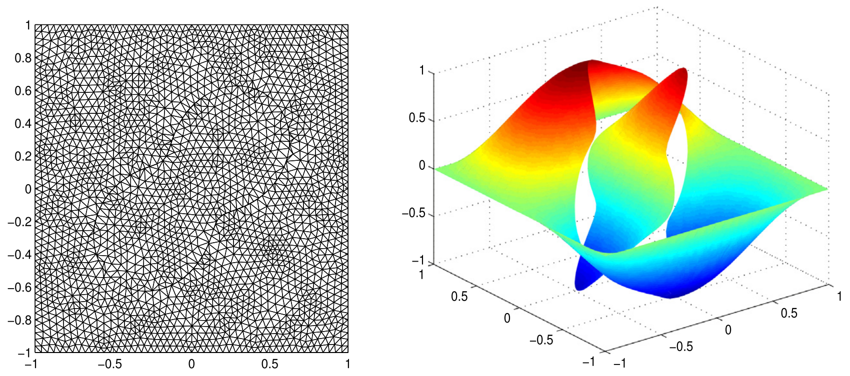

Fig. 3. Plots of Example 2. Left: A triangulation matched with the interface; Right: numerical solution ($T = 5E-2$).

Table 3
Convergence rates in time for the fully implicit SIPG scheme.

Time steps	$\|error\|_{L^2(H^1)}$	*order*	$\|error\|_{L^2(L^2)}$	*order*
$\triangle t$	$9.6313E-02$	–	$2.5278E-02$	–
$\triangle t/2$	$5.2597E-02$	0.873	$1.3766E-02$	0.877
$\triangle t/4$	$2.7068E-02$	0.958	$6.9852E-03$	0.979
$\triangle t/8$	$1.4469E-02$	0.904	$3.5229E-03$	0.988

Table 4
Convergence rates in space for the fully implicit SIPG scheme, $\max\{h\} = 4.1595E-02$.

Mesh size	$\|error\|_{L^2(H^1)}$	*order*	$\|error\|_{L^2(L^2)}$	*order*
$\max\{h\}$	$1.0987E-04$	–	$5.3224E-06$	–
$\max\{h\}/2$	$2.6129E-05$	2.072	$6.4994E-07$	3.034
$\max\{h\}/4$	$8.3275E-06$	1.650	$1.2748E-07$	2.350

where the parameter m determines the number of leaves of the core region Ω_1. We solve the interface problem by the SIPG scheme with piecewise linear elements on a locally refined mesh matched the interface with the parameter $(m, b) = (2, \frac{1}{4})$ (see Fig. 3 Left). A analytical solution with homogeneous Dirichlet boundary condition is constructed by

$$u(x, y, t) = \begin{cases} \sin(2x)\cos(2y)\cos(t), & \text{in } \Omega_1, \\ \cos(\frac{\pi}{2}x)\sin(\pi y)\cos(t), & \text{in } \Omega_2, \end{cases}$$

and the corresponding time-dependent jump conditions can be given by the analytical solution.

We choose a locally refined mesh of size $h = 2.0797E-02$ matched with the two-leave interface and the initial time step $\triangle t = 2E-2$ for a computation to the final time $T = 1$. In the temporal direction, Table 3 demonstrates that the fully implicit SIPG scheme has a first order convergence rate in time.

We employ a class of uniformly refined quasi-uniform meshes to compute convergence order in space. The maximum size of the initial mesh is $\max\{h\} = 4.1595E-02$. With the use of piecewise linear elements on refined meshes matched with the two-leave interface, the time step $\triangle t = 1E-5$ and the final time $T = 5E-2$ are considered. From Table 4, we observe that our method in the spacial direction has a second order convergence rate with respect to mesh sizes in $L^2(H^1)$ and $L^2(L^2)$ norms. Furthermore, on the finest mesh, Fig. 3 Right illustrates the numerical solution without oscillations.

7. Conclusions

New FEMs are developed for solving parabolic interface problems with discontinuous coefficients on a bounded convex polyhedral domain and with jump conditions along a C^2 smooth interface. We analyze the space semidiscrete schemes based on SIPG approximations using discontinuous piecewise linear functions. Error estimates in the broken L^2, H^1 and $L^2(H^1)$ norms are proved for smooth initial data with respect to low regularity solutions, which only are piecewise $H^{1+s}(\mathcal{T}_h)$ smooth with $0 < s \leq 1$. We note that the convergence rates in the proposed approximations depend on the indexes of the global regularity and the local regularity. Furthermore, we analyze the fully discrete SIPG schemes based on the first-order Euler backward finite difference discretization for the time derivative. Such SIPG schemes are shown to be unconditionally stable, and have the nearly optimal $L^2(L^2)$ and $L^2(H^1)$ error estimates, even when the regularity of analytic solutions is low on the whole domain. Numerical experiments for two-dimensional parabolic interface problems confirm the theoretical convergence rates.

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Appendix A. Regularization of (1)

The interface equation (1)–(2) can be transferred into the model problem (3)–(4) by using the following steps.

Step 1. In the convex subdomain Ω_1 with the boundary $\partial\Omega_1 = \Gamma_D$, we solve a homogeneous elliptic problem to find $\tilde{v}_1$ such that

$$\begin{cases} -\nabla\cdot(k(x)\nabla\widetilde{v_1}) = 0, & \text{in } \Omega_1, \\ \widetilde{v_1} = -\phi(x,0), & \text{on } \Gamma_D. \end{cases} \tag{A.1}$$

Step 2. We solve v_1 satisfying a homogeneous parabolic problem defined in $\Omega_1 \times (0,T]$

$$\begin{cases} \dfrac{\partial v_1}{\partial t} - \nabla\cdot(k(x)\nabla v_1) = 0, & \text{in } \Omega_1\times(0,T], \\ v_1(x,0) = \widetilde{v_1}(x), & \text{in } \Omega_1, \\ v_1 = -\phi(x,t), & \text{on } \Gamma_D\times(0,T]. \end{cases} \tag{A.2}$$

Step 3. Assume that $w := u + v$ and

$$v := \begin{cases} v_1, & \text{in } \Omega_1\times(0,T] \\ 0, & \text{in } \Omega_2. \end{cases}$$

Setting the terms

$$u_0(x) := w_0(x) - \widetilde{v_1}(x), \tag{A.3}$$

$$\psi := \tilde{\psi}(x,t) + \frac{\partial v_1}{\partial \mathbf{n}}\Big|_{\Gamma_D}, \tag{A.4}$$

then by a simple calculation, we see that u satisfies (3) with the jump conditions in (4).

Appendix B. Proof of Theorem 5.3

To prove Theorem 5.3, we first consider the following auxiliary problem: Let $w_h^N = 0$. From $n = N, N-1, \ldots, 1$, find $w_h^{n-1} \in V_{hk}^n$ such that

$$-(\widetilde{\nabla} w_h^n, v_h) + a_\lambda(w_h^{n-1}, v_h) = (\bar{u}^n - u_h^n, v_h), \quad \forall v_h \in V_{hk}^n, \tag{B.1}$$

where $\lambda = -1$.

We have the following critical lemmas.

Lemma B.1. *It holds*

$$C\|w_h^n\|_{DG}^2 + \sum_{n=1}^{N} \Delta t\|\widetilde{\nabla} w_h^n\|_{0,\mathcal{T}_h}^2 \le \sum_{n=1}^{N}\Delta t\|\bar{u}^n - u_h^n\|_{0,\mathcal{T}_h}^2. \tag{B.2}$$

Proof. Taking $v_h = -\Delta t\widetilde{\nabla} w_h^n$ in (B.1) with $w_h^N = 0$ gives

$$\Delta t\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}^2 + a_{-1}(w_h^{n-1}, w_h^{n-1}) = a_{-1}(w_h^{n-1}, w_h^n) - (\bar{u}^n - u_h^n, \Delta t\widetilde{\nabla} w_h^n).$$

By coercivity of $a_{-1}(\cdot,\cdot)$, it follows

$$\begin{aligned} \Delta t\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}^2 + \alpha_0\left\|w_h^{n-1}\right\|_{DG}^2 &\le \alpha_1\left\|w_h^{n-1}\right\|_{DG}\left\|w_h^n\right\|_{DG} \\ &\quad + (\Delta t)^{\frac{1}{2}}\left\|\bar{u}^n - u_h^n\right\|_{0,\mathcal{T}_h}\left\|(\Delta t)^{\frac{1}{2}}\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}, \end{aligned}$$

which infers from the Cauchy–Schwarz inequality

$$\begin{aligned} &\frac{\Delta t}{2\alpha_1}\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}^2 + \frac{\alpha_0}{\alpha_1}\left\|w_h^{n-1}\right\|_{DG}^2 \\ &\le \left\|w_h^{n-1}\right\|_{DG}\left\|w_h^n\right\|_{DG} + \frac{\Delta t}{2\alpha_1}\left\|\bar{u}^n - u_h^n\right\|_{0,\mathcal{T}_h}^2 \\ &\le \epsilon\left\|w_h^{n-1}\right\|_{DG}^2 + \frac{1}{4\epsilon}\left\|w_h^n\right\|_{DG}^2 + \frac{\Delta t}{2\alpha_1}\left\|\bar{u}^n - u_h^n\right\|_{0,\mathcal{T}_h}^2, \quad \forall \epsilon > 0, \end{aligned}$$

equivalently,

$$\frac{\triangle t}{2\alpha_1}\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}^2+\left(\frac{\alpha_0}{\alpha_1}-\epsilon\right)\left\|w_h^{n-1}\right\|_{DG}^2 \le \frac{1}{4\epsilon}\left\|w_h^n\right\|_{DG}^2+\frac{\triangle t}{2\alpha_1}\left\|\bar{u}^n-u_h^n\right\|_{0,\mathcal{T}_h}^2. \tag{B.3}$$

Thanks to $0<\frac{\alpha_0}{\alpha_1}\le 1$, it is easy to see that $\frac{\alpha_0}{\alpha_1}-\epsilon\le\frac{1}{4\epsilon}$ holds for any $\epsilon>0$. We set

$$C_0=\frac{\alpha_0}{\alpha_1}-\epsilon,\quad C_1=\frac{1}{4\epsilon},\quad (C_0\le C_1).$$

Summing over n from 1 to N for (B.3) and noting that $w_h^N=0$, we have

$$\begin{aligned}&\frac{1}{2\alpha_1}\sum_{n=1}^{N}\triangle t\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}^2+C_0\left\|w_h^0\right\|_{DG}^2\\ &\le (C_1-C_0)\sum_{n=1}^{N-1}\left\|w_h^n\right\|_{DG}^2+\frac{1}{2\alpha_1}\sum_{n=1}^{N}\triangle t\left\|\bar{u}^n-u_h^n\right\|_{0,\mathcal{T}_h}^2.\end{aligned} \tag{B.4}$$

Thus, applying discrete Gronwall's lemma leads to the conclusion. □

Lemma B.2. *It holds*

$$\sum_{n=1}^{N}\triangle t\left\|w_h^{n-1}\right\|_{DG,\Omega^*}^2\le C\max\left\{h|\log(h)|,\,h^{2(s-\epsilon)}\right\}\sum_{n=1}^{N}\triangle t\left\|\bar{u}^n-u_h^n\right\|_{0,\mathcal{T}_h}^2. \tag{B.5}$$

Proof. Recall that $\Omega^*=\cup_{E\in\mathcal{T}_h^*}E$ and $\mathrm{meas}(\Omega^*)\le Ch$. For $n=N,N-1,\ldots,1$, we define $w^{n-1}\in H_0^1(\Omega)\cap V^{1+s}(\mathcal{T}_h)$ to the solution of the following elliptic interface problem:

$$a_{-1}(w^{n-1},v)=(\bar{u}^n-u_h^n,v)+(\widetilde{\nabla} w_h^n,v),\qquad \forall\, v\in H^1(\Omega)\cap V^{1+s}(\mathcal{T}_h), \tag{B.6}$$

which corresponds to the following jump conditions

$$[w^{n-1}]=0,\quad \left[k\frac{\partial u}{\partial \mathbf{n}}\right]=0,\quad \text{across the interface } \Gamma.$$

Then by the regularity of solutions of interface elliptic equations, we have that $w^{n-1}\in X$ and

$$\|w^{n-1}\|_X\le C\left(\left\|\bar{u}^n-u_h^n\right\|_{0,\mathcal{T}_h}+\left\|\widetilde{\nabla} w_h^n\right\|_{0,\mathcal{T}_h}\right). \tag{B.7}$$

In fact, $w_h^{n-1}\in V_{hk}^n$ is the DG finite element approximation of w^{n-1} defined in (B.6). Analogously to [33, Theorem 3.2], taking $f=\bar{u}^n-u_h^n+\widetilde{\nabla} w_h^n$ results in

$$\begin{aligned}\left\|w^{n-1}-w_h^{n-1}\right\|_{DG}&\le C(h^{s-\epsilon}\|w^{n-1}\|_{V^{1+s}(\mathcal{T}_h)}+h\|f\|_{0,\mathcal{T}_h})\\ &\le C(h^{s-\epsilon}\|w^{n-1}\|_{V^{1+s}(\mathcal{T}_h)}+h\|\bar{u}^n-u_h^n\|_{0,\mathcal{T}_h}+h\|\widetilde{\nabla} w_h^n\|_{0,\mathcal{T}_h}),\end{aligned}$$

which with Lemma B.1 implies

$$\sum_{n=1}^{N}\triangle t\left\|w^{n-1}-w_h^{n-1}\right\|_{DG}^2\le Ch^{2(s-\epsilon)}\|w^{n-1}\|_{L^2(0,T;V^{1+s}(\mathcal{T}_h))}^2+Ch^2\sum_{n=1}^{N}\triangle t\|\bar{u}^n-u_h^n\|_{0,\mathcal{T}_h}.$$

By the triangle inequality and the continuity of w^{n-1}, it follows

$$\begin{aligned}\sum_{n=1}^{N}\triangle t\left\|w_h^{n-1}\right\|_{DG,\Omega^*}^2&\le\sum_{n=1}^{N}\triangle t\left(\left\|w_h^{n-1}-w^{n-1}\right\|_{DG,\Omega^*}^2+\left\|w^{n-1}\right\|_{DG,\Omega^*}^2\right)\\ &\le Ch^{2(s-\epsilon)}\|w^{n-1}\|_{L^2(0,T;V^{1+s}(\mathcal{T}_h))}^2+Ch^2\sum_{n=1}^{N}\triangle t\|\bar{u}^n-u_h^n\|_{0,\mathcal{T}_h}^2\\ &\quad+\sum_{n=1}^{N}\triangle t\left\|w^{n-1}\right\|_{H^1(\Omega^*)}^2.\end{aligned} \tag{B.8}$$

To estimate the last term $\sum_{n=1}^{N}\triangle t\left\|w^{n-1}\right\|^2_{H^1(\Omega*)}$, by using [21, (2.36)], (B.7) and Lemma B.1, we obtain

$$\begin{aligned}\sum_{n=1}^{N}\triangle t\left\|w^{n-1}\right\|^2_{H^1(\Omega*)} &\le Ch|\log h|\sum_{n=1}^{N}\triangle t\left\|w^{n-1}\right\|^2_X\\ &\le Ch|\log h|\sum_{n=1}^{N}\triangle t\|\bar{u}^n-u_h^n\|^2_{0,\mathcal{T}_h},\end{aligned}$$

which with (B.8) implies (B.5). This concludes the proof. □

Now we present the proof of Theorem 5.3 as follows.

Proof of Theorem 5.3. By the triangle inequality, we have

$$\left(\int_0^T\|u-u_h^n\|^2_{0,\mathcal{T}_h}dt\right)^{\frac{1}{2}}\le C\triangle t\|u_t\|_{L^2(0,T;L^2(\Omega))}+\left(\sum_{n=1}^{N}\triangle t\|\bar{u}^n-u_h^n\|^2_{0,\mathcal{T}_h}\right)^{\frac{1}{2}},$$

which means we shall just bound the second term on the right hand.

Taking $v_h=\triangle t(R_h\bar{u}^n-u_h^n)$ in (B.1), we get

$$\begin{aligned}\triangle t\|\bar{u}^n-u_h^n\|^2_{0,\mathcal{T}_h} &= \triangle t(\bar{u}^n-u_h^n,\bar{u}^n-R_h\bar{u}^n)+\triangle t(\bar{u}^n-u_h^n,R_h\bar{u}^n-u_h^n)\\ &=\triangle t(\bar{u}^n-u_h^n,\bar{u}^n-R_h\bar{u}^n)-\triangle t(\widetilde{\nabla}w_h^n,R_h\bar{u}^n-u_h^n)\\ &\quad+\triangle t a_\lambda(w_h^{n-1},R_h\bar{u}^n-u_h^n)\\ &=\triangle t(\bar{u}^n-u_h^n,\bar{u}^n-R_h\bar{u}^n)+\triangle t(-\widetilde{\nabla}w_h^n,R_h\bar{u}^n-u^n)\\ &\quad+\triangle t(-\widetilde{\nabla}w_h^n,R_h\bar{u}^n-u_h^n)+\triangle t\big(a_\lambda(w_h^{n-1},\bar{u}^n-u_h^n)\big),\end{aligned}\tag{B.9}$$

where $\lambda=-1$ and we have used the definition of the operator R_h and the symmetry of $a_{-1}(\cdot,\cdot)$ in the last inequality. By summing over n from 1 to N, (B.9) is rewritten as

$$\begin{aligned}\sum_{n=1}^{N}\triangle t\|\bar{u}^n-u_h^n\|^2_{0,\mathcal{T}_h} &= \sum_{n=1}^{N}\triangle t(\bar{u}^n-u_h^n,\bar{u}^n-R_h\bar{u}^n)+\sum_{n=1}^{N}\triangle t(-\widetilde{\nabla}w_h^n,R_h\bar{u}^n-u^n)\\ &\quad+\sum_{n=1}^{N}\triangle t(-\widetilde{\nabla}w_h^n,R_h\bar{u}^n-u_h^n)+\sum_{n=1}^{N}\triangle t\big(a_\lambda(w_h^{n-1},\bar{u}^n-u_h^n)\big)\\ &\equiv: T_1+T_2+T_3+T_4.\end{aligned}\tag{B.10}$$

Before estimating the four terms in (B.10), we need to rewrite T_3 and T_4 appropriately. We first multiply the first equation of (3) by any $v\in(V^{1+s}(\mathcal{T}_{h,1})\cup V^{1+s}(\mathcal{T}_{h,2}))\cap H_0^1(\Omega)$, then integrate the resultant equation over $\Omega\times I^n$ and apply the conditions in (3) as well as the interface conditions to get for $n=1,2,\ldots,N$,

$$(\widetilde{\nabla}u^n,v)+a_{-1}(\bar{u}^n,v)=(\bar{f}^n,v)+(\bar{\psi}^n,v)_\Gamma,\quad\forall\, v\in(V^{1+s}(\mathcal{T}_{h,1})\cup V^{1+s}(\mathcal{T}_{h,2}))\cap H_0^1(\Omega).\tag{B.11}$$

Taking the test function w_h^{n-1} in (47) and (B.11), then subtracting the two equations and summing the resultant equations over n, we come to

$$\begin{aligned}&\sum_{n=1}^{N}\triangle t(\bar{u}^n-u_h^n,w_h^{n-1})+\sum_{n=1}^{N}\triangle t\big(a_{-1}(\bar{u}^n,w_h^{n-1})-a_{-1}(u_h^n,w_h^{n-1})\big)\\ &=\sum_{n=1}^{N}\triangle t(\bar{f}^n-f^n,w_h^{n-1})+\sum_{n=1}^{N}\triangle t\big((\bar{\psi}^n,w_h^{n-1})_\Gamma-(\bar{\psi}_h^n,w_h^{n-1})_{\Gamma_h}\big).\end{aligned}\tag{B.12}$$

Applying

$$\sum_{n=1}^{N}(a_n-a_{n-1})b_n=a_Nb_N-a_0b_0-\sum_{n=1}^{N}a_{n-1}(b_n-b_{n-1})$$

to T_3 with $a_n = w_h^n$ and $b_n = u^n - u_h^n$, we obtain from (B.12) that

$$\begin{aligned} & T_3 + T_4 \\ & = (u_0 - Q_h u_0, w_h^0) + \sum_{n=1}^N \triangle t(\bar{f}^n - f^n, w_h^{n-1}) + \sum_{n=1}^N \triangle t\big((\bar{\psi}^n, w_h^{n-1})_\Gamma - (\bar{\psi}_h^n, w_h^{n-1})_{\Gamma_h}\big) \\ & \equiv: T_5 + T_6 + T_7, \end{aligned} \tag{B.13}$$

where we have used $w_h^n = 0$ and $u_h^0 = Q_h u_0$.

Now we estimate the terms T_i ($i = 1, 2, 5, 6, 7$). First, analogously to Lemma 3.6, one can obtain

$$\left\| \bar{u}^n - R_h \bar{u}^n \right\|_{0,\mathcal{T}_h} \le Ch^{s-2\epsilon} M_s(h, \bar{u}^n),$$

which can be used to bound the term T_1

$$\begin{aligned} |T_1| & \le \sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h} \left\| \bar{u}^n - R_h \bar{u}^n \right\|_{0,\mathcal{T}_h} \\ & \le Ch^{2(s-\epsilon)} \sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h} \left\| \bar{u}^n \right\|_{V^{1+s}(\mathcal{T}_{h,1}) \cup V^{1+s}(\mathcal{T}_{h,2})} \\ & \le Ch^{2(s-\epsilon)} \Big(\sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h}^2 \Big)^{\frac{1}{2}} \Big(\int_0^T \left\| \bar{u}^n \right\|_{V^{1+s}(\mathcal{T}_h)}^2 dt \Big)^{\frac{1}{2}}. \end{aligned} \tag{B.14}$$

Similarly, using Lemma 3.6 and Cauchy–Schwarz inequality leads to

$$\begin{aligned} |T_2| & \le \Big| \sum_{n=1}^N \triangle t(\widetilde{\nabla} w_h^n, R_h \bar{u}^n - \bar{u}^n) \Big| + \Big| \sum_{n=1}^N \triangle t(\widetilde{\nabla} w_h^n, \bar{u}^n - u^n) \Big| \\ & \le Ch^{2(s-\epsilon)} \sum_{n=1}^N \triangle t \left\| \widetilde{\nabla} w_h^n \right\|_{0,\mathcal{T}_h} \left\| \bar{u}^n \right\|_{V^{1+s}(\mathcal{T}_h)} + \sum_{n=1}^N \triangle t \left\| \widetilde{\nabla} w_h^n \right\|_{0,\mathcal{T}_h} \left\| \bar{u}^n - u^n \right\|_{0,\Omega} \\ & \le C(\triangle t + h^{2(s-\epsilon)}) \Big(\sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h}^2 \Big)^{\frac{1}{2}} \Big(\int_0^T \big(\|u\|_{V^{1+s}(\mathcal{T}_h)}^2 + \|u_t\|_{0,\Omega}^2 \big) dt \Big)^{\frac{1}{2}}, \end{aligned} \tag{B.15}$$

where in the last inequality we have used Lemma B.1 and the fact that

$$\sum_{n=1}^N \triangle t^{\frac{1}{2}} \left\| \bar{u}^n \right\|_{V^{1+s}(\mathcal{T}_h)} \le C \int_0^T \|u\|_{V^{1+s}(\mathcal{T}_h)} dt,$$

and

$$\sum_{n=1}^N \triangle t^{\frac{1}{2}} \left\| \bar{u}^n - u^n \right\|_{0,\Omega} \le C \int_0^T \|u_t\|_{0,\Omega} dt.$$

By the estimate of the operator Q_h, Cauchy–Schwarz inequality and (B.2), we get

$$\begin{aligned} |T_5| + |T_6| & \le \|u_0 - Q_h u_0\|_{0,\mathcal{T}_h} \left\| w_h^0 \right\|_{0,\mathcal{T}_h} + \sum_{n=1}^N \triangle t \left\| \bar{f}^n - f^n \right\|_{0,\mathcal{T}_h} \left\| w_h^{n-1} \right\|_{0,\mathcal{T}_h} \\ & \le Ch^{2(s-\epsilon)} \|u_0\|_{V^{1+s}(\mathcal{T}_h)} \Big(\sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h}^2 \Big)^{\frac{1}{2}} \\ & \quad + C\triangle t \Big(\int_0^T \|f_t\|_{0,\Omega}^2 \Big)^{\frac{1}{2}} \Big(\sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h}^2 \Big)^{\frac{1}{2}} \\ & \le C\big(\triangle t + h^{2(s-\epsilon)}\big) \Big(\sum_{n=1}^N \triangle t \left\| \bar{u}^n - u_h^n \right\|_{0,\mathcal{T}_h}^2 \Big)^{\frac{1}{2}} \Big(\|u_0\|_{V^{1+s}(\mathcal{T}_h)}^2 + \int_0^T \|f_t\|_{0,\Omega}^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{B.16}$$

By Lemmas 5.1 and B.2, the term T_7 can be bounded by

$$\begin{aligned}|T_7| &\le C\sum_{n=1}^{N}\triangle t h^{\frac{3}{2}}\|\bar{\psi}^n\|_{H^2(\Gamma)}\left\|w_h^{n-1}\right\|_{DG,\Omega^*}\\ &\le Ch^{\frac{3}{2}}\Big(\sum_{n=1}^{N}\triangle t\left\|w_h^{n-1}\right\|_{DG,\Omega^*}\Big)^{\frac{1}{2}}\Big(\int_0^T\|\psi\|^2_{H^2(\Gamma)}dt\Big)^{\frac{1}{2}}\\ &\le Ch^{\frac{3}{2}}\max\{h^{\frac{1}{2}}|\log(h)|^{\frac{1}{2}}, h^{(s-\epsilon)}\}\Big(\sum_{n=1}^{N}\triangle t\left\|\bar{u}^n-u_h^n\right\|^2_{0,\mathcal{T}_h}\Big)^{\frac{1}{2}}\Big(\int_0^T\|\psi\|^2_{H^2(\Gamma)}dt\Big)^{\frac{1}{2}}.\end{aligned}\tag{B.17}$$

Finally, Theorem 5.3 follows immediately from (B.10), (B.13) and (B.14)–(B.17). □

Appendix C. Proof of Theorem 5.4

Before proving Theorem 5.4, we first need to introduce the following auxiliary discrete dual problem: Let $z_h^N = 0$. For $n = N, N-1, \ldots, 1$, find $z_h^{n-1} \in V_{hk}^n$ such that

$$-(\widetilde{\nabla} z_h^n, v_h) + a_{-1}(z_h^{n-1}, v_h) = \big(\nabla\big(\bar{u}^n - u_h^n\big), \nabla v_h\big), \quad \forall\, v_h \in V_{hk}^n. \tag{C.1}$$

Clearly, it has a unique solution $z_h^n \in V_{hk}^n$ for each $N, N-1, \ldots, 1$. Then, we prove the following stability results for the solution z_h^n to problem (C.1).

Lemma C.1. *It holds*

$$\|z_h^0\|^2_{0,\mathcal{T}_h} + \sum_{n=1}^{N}\triangle t\left\|z_h^{n-1}\right\|^2_{DG} \le C\sum_{n=1}^{N}\triangle t\left\|\nabla(\bar{u}^n - u_h^n)\right\|^2_{0,\mathcal{T}_h}, \tag{C.2}$$

$$\sum_{n=1}^{N}\triangle t\left\|\widetilde{\nabla} z_h^{n-1}\right\|^2_{H^{-1}(\mathcal{T}_h)} \le C\sum_{n=1}^{N}\triangle t\left\|\nabla(\bar{u}^n - u_h^n)\right\|^2_{0,\mathcal{T}_h}. \tag{C.3}$$

Proof. Taking $v_h = z_h^{n-1}$ in (C.1) gives

$$(-\widetilde{\nabla} z_h^n, z_h^{n-1}) + a_{-1}(z_h^{n-1}, z_h^{n-1}) = \big(\nabla\big(\bar{u}^n - u_h^n\big), \nabla z_h^{n-1}\big). \tag{C.4}$$

The following bounds are valid

$$\begin{aligned}\big(-\widetilde{\nabla} z_h^n, z_h^{n-1}\big) &= \frac{1}{2\triangle t}\Big(\left\|z_h^{n-1}\right\|^2_{0,\mathcal{T}_h} - \left\|z_h^n\right\|^2_{0,\mathcal{T}_h} + \left\|z_h^n - z_h^{n-1}\right\|^2_{0,\mathcal{T}_h}\Big),\\ a_{-1}\big(z_h^{n-1}, z_h^{n-1}\big) &\ge \alpha_0\|z_h^n\|^2_{DG},\\ \big(\nabla\big(\bar{u}^n - u_h^n\big), \nabla z_h^{n-1}\big) &\le \left\|\nabla\big(\bar{u}^n - u_h^n\big)\right\|_{0,\mathcal{T}_h}\left\|\nabla z_h^{n-1}\right\|_{0,\mathcal{T}_h}\\ &\le \frac{1}{2\alpha_0}\left\|\nabla\big(\bar{u}^n - u_h^n\big)\right\|^2_{0,\mathcal{T}_h} + \frac{\alpha_0}{2}\left\|\nabla z_h^{n-1}\right\|^2_{0,\mathcal{T}_h}.\end{aligned}$$

Using the above estimates into (C.4) leads to

$$\left\|z_h^{n-1}\right\|^2_{0,\mathcal{T}_h} + \left\|z_h^n - z_h^{n-1}\right\|^2_{0,\mathcal{T}_h} + \alpha_0\triangle t\|z_h^n\|^2_{DG} \le \left\|z_h^n\right\|^2_{0,\mathcal{T}_h} + \frac{\triangle t}{\alpha_0}\left\|\nabla\big(\bar{u}^n - u_h^n\big)\right\|^2_{0,\mathcal{T}_h}.$$

Summing over n from 1 to N and using $z_h^n = 0$ concludes the proof of (C.2).

Next, we show (C.3). Using (32), the fact that $\|Q_h\phi\|_{DG} = \|\nabla(Q_h\phi)\|_{0,\mathcal{T}_h}$, and the definition of z_h^n, we get

$$\begin{aligned}\|\widetilde{\nabla} z_h^{n-1}\|^2_{H^{-1}(\mathcal{T}_h)} &= \sup_{\phi\in H_0^1(\mathcal{T}_h),\phi\neq 0} \frac{(\widetilde{\nabla} z_h^{n-1},\phi)}{\|\phi\|_{H^1(\mathcal{T}_h)}} = \sup_{\phi\in H_0^1(\mathcal{T}_h),\phi\neq 0} \frac{(\widetilde{\nabla} z_h^{n-1},Q_h\phi)}{\|\phi\|_{H^1(\mathcal{T}_h)}} \\ &\leq \sup_{\phi\in H_0^1(\mathcal{T}_h),\phi\neq 0} \frac{\left|a_{-1}(z_h^{n-1},Q_h\phi)\right| + \left|\left(\nabla(\bar{u}^n-u_h^n),\nabla Q_h\phi\right)\right|}{\|\phi\|_{H^1(\mathcal{T}_h)}} \\ &\leq C\left(\|z_h^{n-1}\|_{DG} + \left\|\nabla(\bar{u}^n-u_h^n)\right\|_{0,\mathcal{T}_h}\right)\frac{\|\nabla Q_h\phi\|_{0,\mathcal{T}_h}}{\|\phi\|_{H^1(\mathcal{T}_h)}} \\ &\leq C\left(\|z_h^{n-1}\|_{DG} + \left\|\nabla(\bar{u}^n-u_h^n)\right\|_{0,\mathcal{T}_h}\right).\end{aligned} \tag{C.5}$$

Summing over n from 1 to N and using (C.2) implies (C.3). □

Now, we prove Theorem 5.4 as follows.

Proof of Theorem 5.4. By the triangle inequality, we have

$$\begin{aligned}\left\|\nabla(u-u_h^n)\right\|_{L^2(0,T;L^2(\mathcal{T}_h))} &\leq \left\|\nabla(u-\bar{u}^n)\right\|_{L^2(0,T;L^2(\mathcal{T}_h))} + \left\|\nabla(\bar{u}^n-u_h^n)\right\|_{L^2(0,T;L^2(\mathcal{T}_h))} \\ &\leq C\Delta t\left(\int_0^T \|u_t\|_Y^2 dt\right)^{\frac{1}{2}} + \left(\sum_{n=1}^N \Delta t\left\|\nabla(\bar{u}^n-u_h^n)\right\|^2_{0,\mathcal{T}_h}\right)^{\frac{1}{2}},\end{aligned}$$

which means we shall just estimate the second term on the right hand.

We write $E(h,\Delta t) \equiv: \sum_{n=1}^N \Delta t\left\|\nabla(\bar{u}^n-u_h^n)\right\|^2_{0,\mathcal{T}_h}$. Using the definition of R_h, taking $v_h = \Delta t(R_h\bar{u}^n - u_h^n) \in V_{hk}^n$ in (C.1) and summing the resultant equations over n, we get

$$\begin{aligned}E(h,\Delta t) &= \sum_{n=1}^N \Delta t\left(\nabla(\bar{u}^n-u_h^n),\nabla(\bar{u}^n-R_h\bar{u}^n)\right) + \sum_{n=1}^N \Delta t\left(\nabla(\bar{u}^n-u_h^n),\nabla(R_h\bar{u}^n-u_h^n)\right) \\ &= \sum_{n=1}^N \Delta t\left(\nabla(\bar{u}^n-u_h^n),\nabla(\bar{u}^n-R_h\bar{u}^n)\right) + \sum_{n=1}^N \Delta t\left(-\widetilde{\nabla} z_h^n, R_h\bar{u}^n-\bar{u}^n\right) \\ &\quad + \sum_{n=1}^N \Delta t\left(-\widetilde{\nabla} z_h^n, \bar{u}^n-u^n\right) + \sum_{n=1}^N \Delta t\left(-\widetilde{\nabla} z_h^n, u^n-u_h^n\right) \\ &\quad + \sum_{n=1}^N \Delta t\left(a_{-1}(z_h^{n-1},\bar{u}^n) - a_{-1}(z_h^{n-1},u_h^n)\right) \\ &\equiv: V_1+V_2+V_3+V_4+V_5.\end{aligned}$$

Analogously to the derivation of (B.12), it follows

$$\begin{aligned}V_4+V_5 &= \left(u_0-Q_hu_0, z_h^0\right) + \sum_{n=1}^N \Delta t\left(\bar{f}^n - f^n, z_h^{n-1}\right) \\ &\quad + \sum_{n=1}^N \Delta t\left((\bar{\psi}^n, z_h^{n-1})_\Gamma - (\bar{\psi}_h^n, z_h^{n-1})_{\Gamma_h}\right) \\ &\equiv: V_6+V_7+V_8.\end{aligned} \tag{C.6}$$

Next, we seek the bounds for the terms $V_1,\ V_2,\ V_3,\ V_6,\ V_7$ and V_8. For V_6+V_7, analogously to (B.16), it is bounded from (C.2)

$$\begin{aligned}|V_6+V_7| &\leq C\left(\Delta t + h^{2(s-\epsilon)}\right)\left(\sum_{n=1}^N \Delta t\left\|\bar{u}^n-u_h^n\right\|^2_{0,\mathcal{T}_h}\right)^{\frac{1}{2}}\left(\|u_0\|^2_{V^{1+s}(\mathcal{T}_h)} + \int_0^T \|f_t\|^2_{0,\Omega}\right)^{\frac{1}{2}} \\ &= C\left(\Delta t + h^{2(s-\epsilon)}\right)E(h,\Delta t)^{\frac{1}{2}}\left(\|u_0\|^2_{V^{1+s}(\mathcal{T}_h)} + \int_0^T \|f_t\|^2_{0,\Omega}\right)^{\frac{1}{2}}.\end{aligned} \tag{C.7}$$

By Lemma 5.1, the term V_8 is bounded by

$$
\begin{aligned}
|V_8| &\le Ch^{\frac{3}{2}} \sum_{n=1}^{N} \triangle t \|\bar{\psi}^n\|_{H^2(\Gamma)} \|z_h^{n-1}\|_{DG} \\
&\le Ch^{\frac{3}{2}} \Big(\sum_{n=1}^{N} \triangle t \|\bar{\psi}^n\|^2_{H^2(\Gamma)} \Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{N} \triangle t \|z_h^{n-1}\|^2_{DG} \Big)^{\frac{1}{2}} \\
&\le Ch^{\frac{3}{2}} E(h, \triangle t)^{\frac{1}{2}} \Big(\int_0^T \|\psi\|^2_{H^2(\Gamma)} dt \Big)^{\frac{1}{2}}.
\end{aligned} \tag{C.8}
$$

For the three terms $V_1,\ V_2$, and V_3, we have from Lemma 3.6

$$
\begin{aligned}
|V_1| &\le Ch^{s-\epsilon} E(h, \triangle t)^{\frac{1}{2}} \Big(\sum_{n=1}^{N} \triangle t \|\bar{u}^n\|^2_{V^{1+s}(\mathcal{T}_h)} \Big)^{\frac{1}{2}} \\
&\le Ch^{s-\epsilon} E(h, \triangle t)^{\frac{1}{2}} \Big(\int_0^T \|u\|^2_{V^{1+s}(\mathcal{T}_h)} dt \Big)^{\frac{1}{2}}.
\end{aligned} \tag{C.9}
$$

By (C.3), the terms are bounded as

$$
\begin{aligned}
|V_2| &\le \sum_{n=1}^{N} \triangle t \|\widetilde{\nabla} z_h^n\|_{H^{-1}(\mathcal{T}_h)} \|R_h \bar{u}^n - \bar{u}^n\|_{H^1(\mathcal{T}_h)} \\
&\le Ch^{s-\epsilon} E(h, \triangle t)^{\frac{1}{2}} \Big(\int_0^T \|u\|^2_{V^{1+s}(\mathcal{T}_h)} dt \Big)^{\frac{1}{2}},
\end{aligned} \tag{C.10}
$$

and

$$
\begin{aligned}
|V_3| &\le \sum_{n=1}^{N} \triangle t \|\widetilde{\nabla} z_h^n\|_{H^{-1}(\mathcal{T}_h)} \|\bar{u}^n - u^n\|_{H^1(\mathcal{T}_h)} \\
&\le C\triangle t E(h, \triangle t)^{\frac{1}{2}} \Big(\int_0^T \|u_t\|^2_Y dt \Big)^{\frac{1}{2}}.
\end{aligned} \tag{C.11}
$$

Consequently, theorem follows from the estimates (C.7)–(C.11). □

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